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Research Paper

COMPUTATION OF NM-POLYNOMIAL AND TOPOLOGICAL INDICES FOR CYCLE RELATED GRAPHS

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ABSTRACT

This paper focuses on the computation of NM-polynomial and several topological indices for cycle related graphs such as Wheel graph, Helm graph and Gear graph. The NM-polynomial is a graph invariant that encodes information about the sub graph structure, which is crucial for understanding the connectivity and combinatorial properties of a graph. We develop formulas and methods for computing the NM-polynomial for specific cycle-related graphs, demonstrating its utility in capturing key graph characteristics.

1. Introduction

Topological indices are numerical descriptors that describe a molecular graph's topology, or structure. In the study of quantitative structure-activity relationships (QSAR) and quantitative structure-property relationships (QSPR), these indices are particularly useful in chemistry [1, 3, 18, 19, 20, 22]. Topological indices are essential for comprehending the

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features and behaviors of cycle-related graphs, which comprise chemical structures with one or more rings. Among different types of indices, degree based indices are derived from the degrees of the vertices of a molecular graph[5, 8, 9, 10, 11, 12]. Firstly, Wiener index is introduced by the chemist Harold Wiener in 1947 [23] and is defined as the sum of the distances between all unordered pairs of vertices of a graph G, that is

$$W(G) = \sum_{u,v \in V(G)} d_G(u,v),$$

where $V(G)$ is the vertex set of G and $d_G(u,v)$ denotes distance from u to v.

In recent years, the application of topological indices has extended beyond traditional QSAR and QSPR studies into areas such as nanotechnology, pharmacology, and materials science. These indices serve as effective mathematical tools for predicting the stability, reactivity, and biological activity of chemical compounds without the need for exhaustive experimental procedures. Furthermore, with the advancement of computational techniques, more sophisticated and generalized forms of topological indices have been developed to capture subtle structural variations within molecular graphs. The study of cycle-containing graphs, such as benzenoid systems and fullerene structures, has especially benefited from these indices, enabling researchers to establish correlations between molecular topology and physicochemical properties with greater precision.

Researchers have recently focused on neighborhood degree-based indices [6, 13, 14, 21]. The relations of some neighborhood degree-based topological indices with the NM-polynomial [4, 7, 15, 17] are shown in Table 1

Definition 1.1. Let G be a graph. Then NM-Polynomial [13, 16] of G is defined as

$$NM(G : x, y) = \sum_{i \le j} m^*_{ij}(G) x^i y^j. \tag{1.1}$$

where $m^*_{ij}(G)$, $i, j \ge 1$ be the number of edges uv of G such that $\delta_u, \delta_v = \{i, j\}$. Here δ_u, δ_v represent the neighborhood degree sum of the vertices u and v in G.

TABLE 1. Expression of TI's using NM-polynomial [2]

TI's	$f(s,t)$	Expression with NM-Polynomial
$M_1^{'}$	$s+t$	$(D_s + D_t)NM(G : s,t)_{s=t=1}$
M_2^*	st	$(D_sD_t)NM(G : s,t)_{s=t=1}$
${}^{nm}M_2$	$\frac{1}{st}$	$(S_sS_t)NM(G : s,t)_{s=t=1}$
$NSDD$	$\frac{s^2+t^2}{st}$	$(D_sS_t + D_tS_s)NM(G : s,t)_{s=t=1}$
NH	$\frac{2}{s+t}$	$2S_sJNM(G; s,t)_{s=t=1}$
NI	$\frac{2}{s+t}$	$S_sJD_sD_tNM(G; s,t)_{s=t=1}$

Where $D_x = x\frac{\partial}{\partial x}[M(G)]$, $D_y = y\frac{\partial}{\partial y}[M(G)]$, $D_xD_y = x_1\frac{\partial}{\partial x}(D_y)$,

$$S_x = \int_0^x \frac{M(G;t,y)}{t}\,dt,\ S_y = \int_0^y \frac{M(G;x,t)}{t}\,dt,\ S_x\,S_y = \int_0^x \frac{S_y(t,y)}{t}dt.$$

2. Results and Discussion

In this section, we obtained NM-polynomial and some topological indices of cycle related graphs such as wheel graph, helm graph and gear graph.

2.1. **Wheel graph.**

Wheel graph is a graph formed by connecting a single universal vertex to all vertices of a cycle.
Let W_n be the wheel graph with $n+1$ number of vertices and $2n$ number of edges. The edge partition of neighborhood degree sequences of W_n is given by

$$E_{(6+n)(3n)} = n,\ E_{(6+n)(6+n)} = n \tag{2.1}$$

Theorem 2.1. *NM-polynomial of wheel graph is given by,*

$$NM(W_n : s,t) = ns^{n+6}\left[t^{3n} + t^{n+6}\right].$$

Proof. By using equation (2.1)

$$\begin{aligned} NM(W_n : s,t) &= m_{(n+6)3n}\, s^{n+6}\, t^{3n} + m(n+6)(n+6)s^{n+6}t^{n+6} \\ &= n\, s^{n+6}\, t^{3n} + n\, s^{n+6}\, t^{n+6} \\ &= ns^{n+6}\left[t^{3n} + t^{n+6}\right]. \end{aligned}$$

□

Theorem 2.2. *Let W_n be the wheel graph , then*

(1) *Neighborhood First Zagreb index,* $M_1^{'}(W_n : s,t) = 6n(n+3)$.
(2) *Neighborhood Second Zagreb index,* $M_2^{*}(W_n : s,t) = 2n(2n^2 + 15n + 18)$.
(3) *Neighborhood Modified Second Zagreb index,* ${}^{nm}M_2(W_n : s,t) = \dfrac{2(2n+3)}{(n+6)^2}$.
(4) *Neighborhood Symmetric division deg index,* $NSDD(W_n : s,t) = \dfrac{(4n+6)^2}{3(n+6)}$.
(5) *Neighborhood Harmonic Index,* $NH(W_n : s,t) = \dfrac{3n(n+3)}{(n+6)(2n+3)}$.
(6) *Neighborhood Inverse Index,* $NI(W_n : s,t) = \dfrac{n(n+6)(5n+3)}{2(2n+3)}$.

Proof. Using equation (2.1) and Table 1

(1) $D_s[NM(W_n)] = s\dfrac{\partial}{\partial s}\left[NM(W_n)\right] = sn(n+6)s^{n+5}\left[t^{3n} + t^{n+6}\right]$

$$= n(n+6)s^{n+6}\left[t^{3n} + t^{n+6}\right]$$

$$D_t\left[NM(W_n)\right] = t\frac{\partial}{\partial s}\left[NM(W_n)\right] = tns^{n+6}\left[3nt^{3n-1} + (n+6)t^{n+5}\right]$$

$$= ns^{n+6}\left[3nt^{3n} + (n+6)t^{n+6}\right]$$

$$M_1'(W_n) = (D_s + D_t)\,[NM(W_n; s,t)]_{s=t=1}$$

$$(D_s + D_t)_{s=t=1} = 6(n^2+3n) = 6n(n+3).$$

(2) $D_sD_t\,[NM(W_n)] = s\frac{\partial}{\partial s}\,[D_t)] = sn(n+6)s^{n+5}\left[3nt^{3n} + (n+6)t^{n+6}\right]$

$$= s\Big[n(n+6)s^{n+5}\left[3nt^{3n} + (n+6)t^{n+6}\right]\Big]$$

$$= 3n^2(n+6)s^{n+6}\,t^{3n} + n(n+6)^2 s^{n+6}\,t^{n+6}$$

$$M_2^*(W_n) = (D_sD_t)\,[NM(W_n; s,t)]_{s=t=1} = 3n^2(n+6) + n(n+6)^2$$

$$= 2n(2n^2+15n+18).$$

(3) $S_s = \displaystyle\int_0^s \frac{NM(W_n; x,t)}{x}\,dx$

$$= \int_0^s nx^{n+5}\left[t^{3n} + t^{n+6}\right]dx$$

$$= n\,\frac{s^{n+6}}{n+6}\Big[t^3n + t^{n+6}\Big]$$

$$S_t = \int_0^t \frac{NM(W_n; s,y)}{y}\,dx$$

$$= \textstyle\int_0^t\, ns^{n+6}\left[y^{3n-1} + y^{n+5}\right]dy$$

$$= n\,s^{n+6}\Big[\frac{t^{3n}}{3n} + \frac{t^{n+6}}{n+6}\Big]$$

$$S_s\,S_t = \int_0^s \frac{S_t(x,t)}{x}\,dx$$

$$= \int_0^s nx^{n+5}\left[\frac{t^{3n}}{3n} + \frac{t^{n+6}}{n+6}\right]dx$$

$$= n\,\frac{s^{n+6}}{n+6}\Big[\frac{t^{3n}}{3n} + \frac{t^{n+6}}{n+6}\Big]$$

$$^{nm}M_2(W_n : s,t) = (S_sS_t)\Big[NM(W_n; s,t)\Big]_{s=t=1}$$

$$= \frac{2(2n+3)}{(n+6)^2}.$$

(4) $NSDD(W_n : s,t) = [D_sS_t + S_sD_t]\,[NM(W_n;s,t)]_{s=t=1}$

$$\begin{aligned} D_sS_t &= s\frac{\partial}{\partial s}(S_t) \\ &= s\frac{\partial}{\partial s}\left[ns^{n+6}\left(\frac{t^{3n}}{3n} + \frac{t^{n+6}}{n+6}\right)\right] \\ &= n(n+6)s^{n+6}\left[\frac{t^{3n}}{3n} + \frac{t^{n+6}}{n+6}\right] \end{aligned}$$

$$\begin{aligned} S_sD_t &= \int_0^s \frac{D_t(x,t)}{x}\,dx \\ &= \int_0^s nx^{n+5}\Big[3nt^{3n} + (n+6)t^{n+6}\Big]dx \\ &= n\frac{s^{n+6}}{n+6}\Big[3nt^{3n} + (n+6)t^{n+6}\Big] \end{aligned}$$

$$\begin{aligned} NSDD(W_n) &= n(n+6)\Big[\frac{1}{3n} + \frac{1}{n+6}\Big] + \frac{n}{n+6}\Big[3n + (n+6)\Big] \\ &= \frac{(4n+6)^2}{3(n+6)}. \end{aligned}$$

(5) $NH(W_n : S,t) = 2S_sJ\Big[NM(W_n : s,t)\Big]_{s=t=1}$

$$\begin{aligned} J[NM(W_n;s,t)] &= NM(W_n;s,s) \\ &= ns^{n+6}\Big[s^{3n} + s^{n+6}\Big] \\ &= n\Big[s^{4n+6} + s^{2n+12}\Big] \end{aligned}$$

$$\begin{aligned} S_sJ\Big[NM(W_n)\Big] &= \int_0^s \frac{J\Big[NM(W_n)\Big]_x}{x}\,dx \\ &= n\int_0^s (x^{4n+5} + x^{2n+11})dx \\ &= n\Big[\frac{s^{4n+6}}{4n+6} + \frac{s^{2n+12}}{2n+12}\Big] \end{aligned}$$

$$\begin{aligned} NH(W_n : S,t) &= 2n\Big[\frac{1}{4n+6} + \frac{1}{2n+12}\Big] \\ &= \frac{3n(n+3)}{(n+6)(2n+3)}. \end{aligned}$$

(6) $NI(W_n : S,t) = S_sJD_sD_t\Big[NM(W_n : s,t)\Big]_{s=t=1}$

$$D_sD_t\Big[NM(W_n)\Big] = 3n^2(n+6)s^{n+6}\,t^{3n} + n(n+6)^2s^{n+6}\,t^{n+6}$$

$$J(D_s\,D_t) = 3n^2\,(n+6)\,s^{4n+6} + n(n+6)^2s^{2n+12}$$

$$S_x\,J(D_s\,D_t) = \int_0^s 3n^2\,(n+6)\,x^{4n+6} + n(n+6)^2x^{2n+11}\,dx$$

$$= 3n^2(n+6)\,\frac{s^{4n+6}}{4n+6} + n(n+6)^2\,\frac{s^{2n+12}}{2n+12}$$

$$NI(W_n:S,t) = 3n^2(n+6)\,\frac{1}{4n+6} + n(n+6)^2\,\frac{1}{2n+12}$$

$$= n(n+6)\Big[\frac{3n}{4n+6} + \frac{n+6}{2(n+6)}\Big]$$

$$= \frac{n(n+6)(5n+3)}{2(2n+3)}.$$

□

2.2. **Helm graph.** The helm graph is the graph obtained from a wheel graph by adjoining a pendant edge at each node of the cycle.
Let H_n be the helm graph having number of edges $3n$ and number of vertices $2n+1$. The neighborhood degree sequence of edge partition of H_n is given below

(2.2) $$E_{(9+n)(4)} = n,\;\; E_{(9+n)(9+n)} = n,\;\; E_{(9+n)(9+n)} = n$$

Theorem 2.3. *NM-polynomial of helm graph is given by,*
$NM(H_n:s,t) = ns^{n+9}\Big[t^{4n} + t^4 + t^{n+9}\Big].$

Proof. By using equation (2.2)

$$NM(H_n:s,t) = m_{(n+9)4n}\,s^{n+9}\,t^{4n} + m_{(n+9)4}\,s^{n+9}\,t^4 + m_{(n+9)(n+9)}\,s^{n+9}\,t^{n+9}$$

$$= n\,s^{n+9}\,t^{4n} + n\,s^{n+9}\,t^4 + n\,s^{n+9}\,t^{n+9}$$

$$= ns^{n+9}\Big[t^{4n} + t^4 + t^{n+9}\Big].$$

□

Theorem 2.4. *Let H_n be the helm graph , then*

(1) *Neighborhood First Zagreb index, $M_1^{'}(H_n:s,t) = 8n(n+5)$.*

(2) *Neighborhood Second Zagreb index, $M_2^{*}(H_n:s,t) = n(5n^2+58n+117)$.*

(3) *Neighborhood Modified Zagreb index,* ${}^{nm}M_2(H_n : s,t) = \dfrac{n^2+14n+9}{4(n+9)^2}$.

(4) *Neighborhood Symmetric division deg index,* $NSDD(H_n : s,t) = \dfrac{n^3+23n^2+155n+133}{4(n+9)}$.

(5) *Neighborhood Harmonic Index,* $NH(H_n : s,t) = \dfrac{n(17n^2+226n+513)}{(5n+9)(n+13)(n+9)}$.

(6) *Neighborhood Inverse Index* $NI(H_n : s,t) = \dfrac{13n^3+335n^2+2151n+1701}{2(5n+9)(n+13)}$.

Proof. Using equation (2.2) and Table 1

(1) $D_s\left[NM(H_n)\right] = s\dfrac{\partial}{\partial s}\left[NM(H_n)\right] = sn(n+9)s^{n+8}\left[t^{4n}+t^4+t^{n+9}\right].$

$$= n(n+9)s^{n+9}\left[t^{4n}+t^4+t^{n+9}\right]$$

$$D_t\left[NM(H_n)\right] = t\frac{\partial}{\partial s}\left[NM(H_n)\right] = tns^{n+9}\left[4nt^{4n-1}+4t^3+(n+9)t^{n+8}\right]$$

$$= ns^{n+9}\left[4nt^{4n}+4t^4+(n+9)t^{n+9}\right]$$

$$M_1'(H_n) = (D_s+D_t)\left[NM(H_n;s,t)\right]_{s=t=1}$$

$$= 8n(n+5).$$

(2) $D_sD_t\left[NM(H_n)\right] = s\dfrac{\partial}{\partial s}\left[D_t)\right] = sn(n+9)s^{n+8}\left[4nt^{4n}+4t^4+(n+9)t^{n+9}\right]$

$$= n(n+9)s^{n+9}\Big[4nt^{4n}+4t^4+(n+9)t^{n+9}\Big]$$

$$M_2^*(H_n) = (D_sD_t)\left[NM(H_n;s,t)\right]_{s=t=1} = n(n+9)[4n+4+(n+9)]$$

$$= 5n^3+58n^2+117.$$

(3) $S_s\,S_t = \displaystyle\int_0^s \frac{S_t(x,t)}{x}\,dx$

$$= n\,\frac{s^{n+9}}{n+9}\Big[\frac{t^{4n}}{4n}+\tfrac{t^4}{4}+\frac{t^{n+9}}{n+9}\Big]$$

$${}^{m}M_2(H_n : s,t) = (S_sS_t)\Big[NM(H_n;s,t)\Big]_{s=t=1}$$

$$= \frac{n}{n+9}\Big[\tfrac{1}{4n}+\tfrac{1}{4}+\tfrac{1}{n+9}\Big]$$

$$= \frac{n^2+14n+9}{4(n+9)^2}.$$

(4) $S_s D_t = \int_0^s \frac{D_t(x,t)}{x}\,dx$

$$= \int_0^s nx^{n+8}\left[4nt^{4n} + 4t^4 + (n+9)t^{n+9}\right]dx$$

$$= n\,\frac{s^{n+9}}{n+9}\Big[4nt^{4n} + 4t^4 + (n+9)t^{n+9}\Big]$$

$$D_s S_t = s\frac{\partial}{\partial s}(S_t)$$

$$= s\frac{\partial}{\partial s}\Big[ns^{n+9}\Big(\frac{t^{4n}}{4n} + \frac{t^4}{4} + \frac{t^{n+9}}{n+9}\Big)\Big]$$

$$= n(n+9)s^{n+9}\Big[\frac{t^{4n}}{4n} + \frac{t^4}{4} + \frac{t^{n+9}}{n+9}\Big]$$

$$NSDD(H_n : s,t) = [D_sS_t + S_sD_t]\,[NM(H_n; s,t)]_{s=t=1}$$

$$= n(n+9)\Big[\frac{1}{4n} + \frac{1}{4} + \frac{1}{n+9}\Big] + \frac{n}{n+9}\Big[4n + 4n + n + 9\Big]$$

$$NSDD(H_n) = \frac{n^3 + 43n^2 + 187n + 81}{4(n+9)}.$$

(5) $J[NM(H_n; s,t)] = NM(H_n; s,s)$

$$= ns^{n+9}\left[s^{4n} + s^4 + s^{n+9}\right]$$

$$= n\left[s^{5n+9} + s^{n+13} + s^{2n+18}\right]$$

$$S_sJ\Big[NM(H_n)\Big] = \int_0^s \frac{J\Big[NM(H_n)\Big]_x}{x}\,dx$$

$$= n\int_0^s (x^{5n+8} + x^{n+12} + x^{2n+17})dx$$

$$= n\Big[\frac{s^{5n+9}}{5n+9} + \frac{s^{n+13}}{n+13} + \frac{s^{2n+18}}{2n+18}\Big]$$

$$NH(H_n : S,t) = 2S_sJ\Big[NM(H_n : s)\Big]_{s=1}$$

$$= 2n\Big[\frac{1}{5n+9} + \frac{1}{n+13} + \frac{1}{2n+18}\Big]$$

$$NH(H_n : S,t) = \frac{n(17n^2 + 226n + 513)}{(5n+9)(n+13)(n+9)}.$$

(6) $NI(H_n : S,t) = S_s J D_s D_t \Big[NM(H_n : s,t)\Big]_{s=t=1}$

$$D_s D_t\left[NM(H_n)\right] = n(n+9)s^{n+9}\Big[4nt^{4n} + 4t^4 + (n+9)t^{n+9}\Big]$$

$$J(D_s\, D_t) = (n+9)\, s^{n+9}\Big[4ns^{4n} + 4s^4 + (n+9)s^{n+9}\Big]$$

$$= (n+9)\Big[4ns^{5n+9} + 4s^{n+13} + (n+9)s^{2n+18}\Big]$$

$$S_s\, J(D_s\, D_t) = \int_0^s (n+9)\, s^{n+9}\Big[4ns^{4n} + 4s^4 + (n+9)s^{n+9}\Big]\, dx$$

$$= (n+9)\Big[4n\,\frac{s^{5n+9}}{5n+9} + 4\,\frac{s^{n+13}}{n+13} + (n+9)\,\frac{s^{2n+18}}{2n+18}\Big]$$

$$NI(H_n : S,t) = (n+9)\Big[\frac{4n}{5n+9} + \frac{4}{n+13} + \frac{(n+9)}{2n+18}\Big]$$

$$NI(H_n : s,t) = \frac{13n^3 + 335n^2 + 2151n + 1701}{2(5n+9)(n+13)}.$$

□

2.3. **Gear graph.** A Gear graph is obtained by adding a vertex to each outer edge of Wheel graph.

Let G_n be the gear graph with $3n$ number of edges and $2n+1$ number of vertices and its edge partition of neighbourhood degree sequence of G_n is

(2.3) $$E_{(n+4)(3n)} = n,\ E_{(n+4)(6)} = 2n$$

Theorem 2.5. *NM-polynomial of gear graph is given by,*

$$NM(G_n : s,t) = ns^{n+4}\Big[t^{3n} + 2t^6\Big].$$

Proof. By using the equation (2.3)

$$NM(G_n : s,t) = m_{(n+4)(3n)}\, s^{n+4}\, t^{3n} + m_{(n+4)(6)}\, s^{n+4}\, t^6$$

$$= n\, s^{n+4}\, t^{3n} + 2ns^{n+4}\, t^6$$

$$= ns^{n+4}\Big[t^{3n} + 2t^6\Big].$$

□

Theorem 2.6. *Let G_n be the gear graph , then*

(1) *Neighborhood First Zagreb index,* $M_1'(G_n : s,t) = 6n(n+4)$.

(2) *Neighborhood Second Zagreb index,* $M_2^*(G_n : s,t) = 3n(n+4)^2$.

(3) *Neighborhood Modified Second Zagreb index,* ${}^{nm}M_2(G_n : s,t) = \dfrac{n+1}{n+4}$.

(4) *Neighborhood Symmetric division deg index,* $NSDD(G_n : s,t) = \dfrac{n^2+14n+4}{3}$.

(5) *Neighborhood Harmonic Index,* $NH(G_n : s,t) = \dfrac{9n(n+2)}{2(n+1)(n+10)}$.

(6) *Neighborhood Inverse Index,* $NI(G_n : s,t) = \dfrac{3n(n+4)(n^2+26n+16)}{4(n+1)(n+10)}$.

Proof. Using equation (2.3) and Table 1

(1) $D_s\left[NM(G_n)\right] = s\dfrac{\partial}{\partial s}\left[NM(G_n)\right] = sn(n+4)s^{n+3}\left[t^{3n}+2t^6\right]$

$$= n(n+4)s^{n+4}\left[t^{3n}+2t^6\right]$$

$$D_t\left[NM(G_n)\right] = t\tfrac{\partial}{\partial s}\left[NM(G_n)\right] = t\,n\,s^{n+4}\Big(3nt^{3n-1}+12t^5\Big)$$

$$= n\,s^{n+4}\Big(3nt^{3n}+12t^6\Big)$$

$$M_1^{'}(G_n) = (D_s+D_t)\left[NM(G_n;s,t)\right]_{s=t=1}$$

$$= 6n(n+4).$$

(2) $D_sD_t\left[NM(G_n)\right] = s\dfrac{\partial}{\partial s}\left(D_t\right) = s\,n\,\Big(3nt^{3n}+12t^6\Big)(n+4)s^{n+3}$

$$= n(n+4)s^{n+4}\Big(3nt^{3n}+12t^6\Big)$$

$$M_2^{*}(G_n) = (D_sD_t)\left[NM(G_n;s,t)\right]_{s=t=1} = n(n+4)(3n+12)$$

$$= 3n(n+4)^2.$$

(3) $S_s\,S_t = \displaystyle\int_0^s \frac{S_t(x,t)}{x}\,dx$

$$= \int_0^s nx^{n+3}\left[\frac{t^{3n}}{3n}+\frac{t^6}{3}\right]dx$$

$$= n\,\frac{s^{n+4}}{n+4}\left[\frac{t^{3n}}{3n}+\frac{t^6}{3}\right]$$

$${}^{nm}M_2(G_n : s,t) = (S_sS_t)\left[NM(H_n;s,t)\right]_{s=t=1}$$

$$= \frac{n}{n+4}\left[\frac{1}{3n}+\frac{1}{3}\right]$$

$$= \frac{n+1}{n+4}.$$

(4) $S_s\,D_t = \displaystyle\int_0^s \frac{D_t(x,t)}{x}\,dx$

$$= \int_0^s nx^{n+3}\Big[3nt^{3n}+12t^6\Big]dx$$

$$= 3n\,\frac{s^{n+4}}{n+4}\Big[nt^{3n} + 4t^{6}\Big]$$

$$D_sS_t = s\frac{\partial}{\partial s}(S_t)$$

$$= s\frac{\partial}{\partial s}\Big[ns^{n+4}\Big(\frac{t^{3n}}{3n} + \frac{t^{6}}{3}\Big)\Big]$$

$$= n(n+4)s^{n+4}\left[\frac{t^{3n}}{3n} + \frac{t^{6}}{3}\right]$$

$$\Big[D_sS_t + S_sD_t\Big]_{s=t=1} = n(n+4)\Big[\frac{1}{3n} + \frac{1}{3} + \frac{3}{n+4}\Big]$$

$$NSDD(G_n : s,t) = \Big[D_sS_t + S_sD_t\Big]\Big[NM(G_n; s,t)\Big]_{s=t=1}$$

$$NSDD(G_n) = \frac{1}{3}(n^2 + 14n + 4).$$

(5) $J[NM(G_n; s,t)] = NM(G_n; s,s)$

$$= ns^{n+4}\left[s^{3n} + 2s^{6}\right]$$

$$= n\Big[s^{4n+4} + 2s^{n+10}\Big]$$

$$S_sJ\Big[NM(G_n)\Big] = \int_0^s \frac{J\Big[NM(G_n)\Big]_x}{x}\,dx$$

$$= n\int_0^s \Big(x^{4n+3} + 2x^{n+9}\Big)dx$$

$$= n\Big[\frac{s^{4n+4}}{4n+4} + \frac{2s^{n+10}}{n+10}\Big]$$

$$NH(G_n : S,t) = 2S_sJ\Big[NM(G_n : s)\Big]_{s=1}$$

$$= 2n\Big[\frac{1}{4n+4} + \frac{2}{n+10}\Big]$$

$$NH(G_n : S,t) = \frac{9n(n+2)}{2(n+1)(n+10)}.$$

(6) $D_sD_t\,[NM(G_n)] = n(n+4)s^{n+4}\Big[3nt^{3n} + 12t^{6}\Big]$

$$J(D_s\,D_t) = n(n+4)s^{n+4}\Big[3ns^{3n} + 12s^{6}\Big]$$

$$= 3n^2(n+4)s^{4n+4} + 12n(n+4)s^{n+10}$$

$$S_s\,J(D_s\,D_t) = \int_0^s \Big[3n^2(n+4)x^{4n+3} + 12n(n+4)x^{n+9}\Big]\,dx$$

$$= 3n^2(n+4)\,\frac{s^{4n+4}}{4n+4} + 12n(n+4)\,\frac{s^{n+10}}{n+10}$$

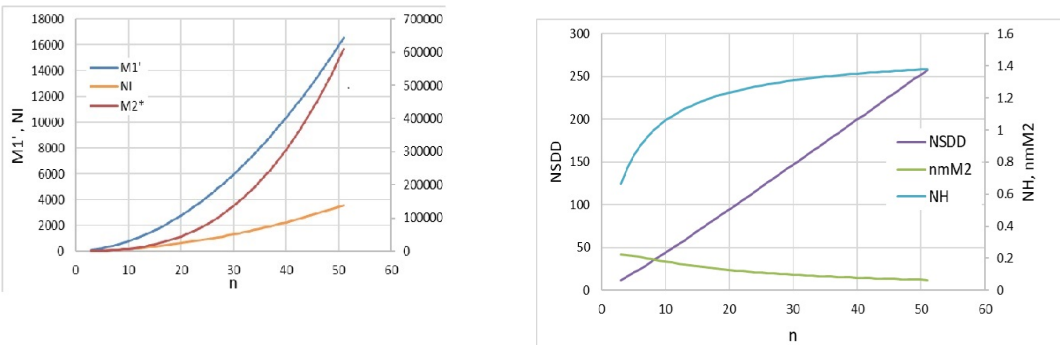
$$NI(G_n : S, t) = \frac{3n^2(n+4)}{4n+4} + \frac{12n(n+4)}{n+10}$$

$$NI(G_n : s, t) = \frac{3n(n+4)(n^2+26n+16)}{4(n+1)(n+10)}.$$

□

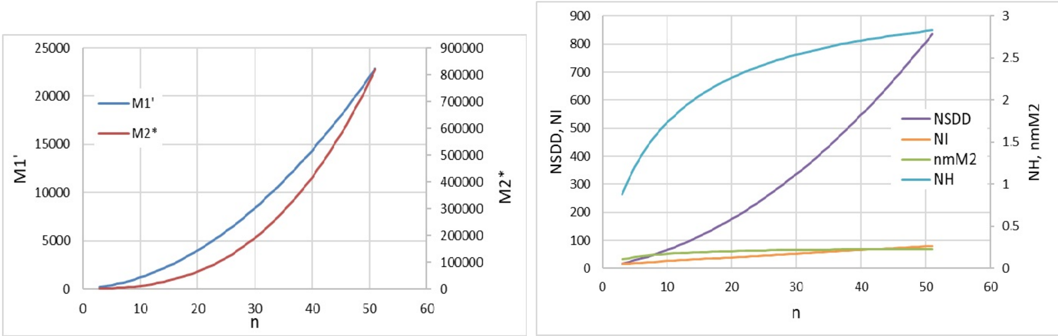
3. Comparison

In all three graphs shown in Figures 1-3, $NSDD$ shows the most substantial and consistent growth, maintaining a steady linear increase without slowing down, but $M_1', M_2, and M_2*$ grow exponentially, with M2* showing the most rapid growth. NH grows rapidly at first but begins to plateau, suggesting that its growth has an upper limit or reaches a steady state. NI grows very slowly in comparison, with a much smaller increase over the same range of n. nmM_2 is nearly flat, indicating a stable, unchanging value throughout the range, suggesting that it is unaffected by the increase in n.



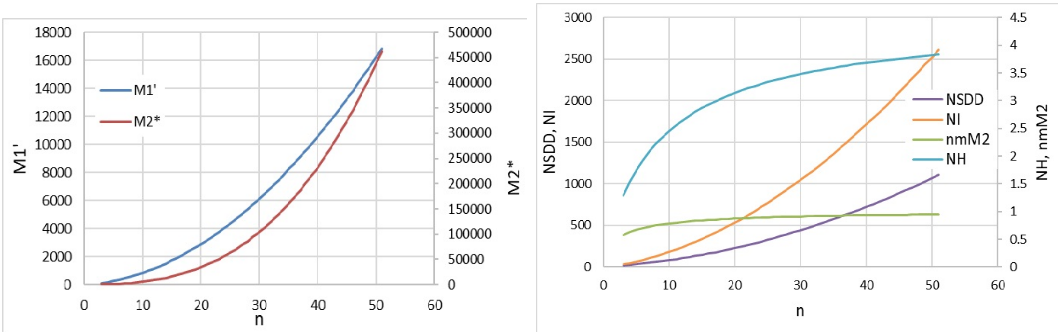
(A) M_1^1, M_2^*, NI (B) $NSDD, nmM_2, NH$

Figure 1. Wheel graph



(A) M_1^1, M_2^*, NI (B) $NSDD, nmM_2, NH$

FIGURE 2. Helm graph



(A) M_1^1, M_2^*, NI (B) $NSDD, nmM_2, NH$

FIGURE 3. Gear graph

4. CONCLUSION

The NM-polynomial and related topological indices derived for cycle-based graphs such as wheel, helm, and gear graphs have broad applications in theoretical chemistry and material science. These descriptors are instrumental in QSAR/QSPR modeling, molecular database indexing, and predicting physicochemical properties of ring-structured compounds. Additionally, they offer valuable insights into structural complexity and connectivity in network science and nanostructure design, demonstrating the wide-reaching utility of the methods developed in this study.

A promising future direction is to extend these methods to dynamic and weighted networks, such as biological interaction networks or evolving chemical reaction networks. Incorporating temporal or weighted connectivity into NM-polynomials could provide deeper insights into time-dependent properties, enabling better modeling of dynamic molecular systems, smart materials, or responsive nanostructures. This would broaden the applicability from static structures to real-world, evolving systems in both chemistry and materials science.

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