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Research Paper

AN OPTIMAL METHOD FOR FINDING COMMON SOLUTIONS TO VARIATIONAL INEQUALITY PROBLEMS VIA SIMULATION FUNCTION

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ABSTRACT

In this paper, we study the existence and uniqueness of common best proximity points for new types of generalized **Z**-contraction pairs, generalized proximal contraction pairs, and generalized **Z**- proximal contraction pairs of non-self mappings defined on a complete metric spaces. Our results improve and generalize some recent findings in the literature. We provide several examples to illustrate the generality of our main results. As an application, we establish sufficient conditions for the existence of unique common solutions to variational inequality problems in Hilbert spaces.

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1. INTRODUCTION AND PRELIMINARIES

When a non-self mapping S has no fixed points, best approximation results provide an approximate solution to the fixed-point equation $Sa = a$ (see [6, 9]). Furthermore, the concept of a best proximity point evolves as a generalization of best approximation. Best proximity point theorems are instrumental in finding an optimal approximate solutions.
Let $S : A \to B$ be a non-self mapping, where A and B are nonempty subsets of a metric space (X, d). If S has no fixed point, we seek an element $a \in A$ that minimizes the distance $d(a, Sa)$. The best proximity point theory ensures the existence of an element $a \in A$ satisfies:

$$d(a, Sa) = d(A, B) = \inf\{d(a, b) : a \in A, b \in B\}.$$

Here, a is called the best proximity point of S. Notably, best proximity theorems also serve as a natural generalization of fixed point theorems since, in the case of a self-mapping, a best proximity point reduces to a fixed point. Many researchers have investigated best proximity point theory in various spaces under different contraction conditions (see [10, 11, 12, 17]). Recently, best proximity point theorems have been studied using the concept of simulation functions introduced by Khojasteh et al. [7](see [13, 14]).

In this work, we extend certain definitions of Tchier et al.[19] for two non-self mappings to establish new common best proximity point theorems in complete metric spaces. In the following, some of the basic theorems of [19] are derived from our main theorems. We explain our main results with several examples. In the end, as applications of the obtained conclusions, we investigate the existence of sufficient conditions for the unique common solutions to variational inequality problems in Hilbert space.
Given two nonempty subsets A and B of a metric space (X, d), the following notions and notations are used in the sequel:

$$\begin{aligned} d(A, B) &= inf\{d(a, b) : a \in A, b \in B\},\\ A_0 &= \{a \in A : d(a, b) = d(A, B), \text{ for some } b \in B\},\\ B_0 &= \{b \in B : d(a, b) = d(A, B), \text{ for some } a \in A\}. \end{aligned}$$

Kirk et al. gave sufficient conditions to ensure that A_0 and B_0 are nonempty sets (see [9]). Raj [18] presented a different aspect in the best proximity point theory by introducing the concept of the P-property.

Definition 1.1. If $A_0 \neq \varnothing$, then the pair (A, B) is said to have the P-property if and only if for any $a_1, a_2 \in A_0$ and $b_1, b_2 \in B_0$,

$$\begin{cases} d(a_1, b_1) = d(A, B) \\ d(a_2, b_2) = d(A, B) \end{cases} \Longrightarrow d(a_1, a_2) = d(b_1, b_2)$$

Using the P-property, Sankar Raj [18] proved an extended version of the Banach contraction principle [2]. We know that for every nonempty subset A of X, the pair (A, A) has the P-property.

Definition 1.2. An element $a \in A$ is said to be a common best proximity point of the non-self mappings $S_1, S_2, ..., S_n : A \to B$ if it satisfies the condition:

$$d(a, S_1a) = d(a, S_2a) = ... = d(a, S_na) = d(A, B).$$

Definition 1.3. A simulation function is a function $\zeta : [0,\infty[\times[0,\infty[\to \mathbb{R}$ satisfying the following conditions:

ζ_1) $\zeta(p,q) < q - p, \forall p, q > 0$

ζ_2) if p_n and q_n are sequences in $(0,\infty)$ such that $p_n < q_n$ for all $n \in \mathbb{N}$ and $\lim\limits_{n\to\infty} p_n = \lim\limits_{n\to\infty} q_n = l > 0$, then

$$\limsup_{n\to\infty} \zeta(p_n, q_n) < 0.$$

We denote the set of all simulation functions by $\mathbf{Z}$.

Remark 1.4. Originally, simulation function was defined by Khojasteh et al. [7] as mapping $\zeta : [0,\infty[\times[0,\infty[\to \mathbb{R}$ satisfying $\zeta(0,0) = 0$ alongside conditions (ζ_1) and (ζ_2) of Definition 1.3. In this paper, a modified definition by Argoubi et al. [1] is used.

Example 1.5. Let $\zeta_\lambda : [0,\infty[\times[0,\infty[\to \mathbb{R}$ be the function defined by $\zeta_\lambda(p,q) = \lambda q - p$, where $\lambda \in [0,1[$. Then ζ_λ is a simulation function.

Theorem 1.6. [7] *Let (X,d) be a complete metric space and $f : X \to X$ be a $\mathbf{Z}$-contraction with respect to ζ, that is,*

$$\zeta(d(fx,fy), d(x,y)) \geq 0, \qquad \forall x, y \in X.$$

Then f has a unique fixed point. Moreover, for every $x_0 \in X$, the Picard sequence $\{f^n x_0\}$ converges to this fixed point.

2. Main Results

We begin our study with the following definition.

Definition 2.1. Let (X,d) be a metric space, and let $S,T : A \to B$ be non-self mappings. We say that $\{S,T\}$ is a generalized $\mathbf{Z}$-contraction pair if there exists a simulation function $\zeta \in \mathbf{Z}$ such that

$$\zeta(d(Sx,Ty), d(x,y)) \geq 0, \qquad \text{for all } x, y \in A.$$

Definition 2.2. Let (X,d) be a metric space, and let $S,T : A \to B$ be non-self mappings. Suppose that $d(u,Sx) = d(v,Ty) = d(A,B)$ for some $u,v,x,y \in A$. Then, we define the following concepts:

i) $\{S,T\}$ is said to be a generalized proximal contraction pair if there exists $\alpha \in [0,1[$ such that

$$d(u,v) \leq \alpha d(x,y).$$

ii) $\{S,T\}$ is said to be a generalized $\mathbf{Z}$-proximal contraction pair if there exists a simulation function $\zeta \in \mathbf{Z}$ such that

$$\zeta(d(u,v), d(x,y)) \geq 0.$$

Remark 2.3. In the preceding definitions, if $S = T$, the notions of generalized $\mathbf{Z}$-contraction, generalized proximal contraction and generalized $\mathbf{Z}$- proximal contraction pair reduce to $\mathbf{Z}$-contraction with respect to ζ (as defined by Khojasteh et al. [7]), proximal contraction and $\mathbf{Z}$-proximal contraction pair of the first kind (as introduced by Tchier et al. [19]), respectively.

Example 2.4. Let $X = \mathbb{R}^2$ be equipped with the Euclidean metric. Consider the sets:

$$A := \{(0, a) : a \in [0, \frac{1}{2}]\} \quad \text{and} \quad B := \{(1, a) : a \in [0, 1]\}.$$

It is easy to see that $d(A, B) = 1$. We define the mappings $S, T : A \to B$ by

$$S(0, x) = T(0, x) = (1, \frac{x}{1+x}), \quad \text{for all} \quad x \in A,$$

and the simulation function $\zeta : [0, \infty[\times[0, \infty[\to \mathbb{R}$ by

$$\zeta(p, q) = \begin{cases} q - \frac{p}{1-p}, & if p \in [0, 1], \\ q - 2p, & otherwise. \end{cases}$$

We can show that $\{S, T\}$ is a generalized **Z**-proximal contraction pair but not a generalized proximal contraction pair.
Suppose $d((0, u), S(0, x)) = d((0, v), T(0, y)) = 1 = d(A, B)$ for some $u, v, x, y \in [0, \frac{1}{2}]$. Then $(x, y) = (\frac{u}{1-u}, \frac{v}{1-v})$, with $u, v \in [0, \frac{1}{3}]$, and hence

$$\begin{aligned} &\zeta\left(d((0, u), (0, v)), d((0, x), (0, y))\right) \\ &= \zeta\left(d((0, u), (0, v)), d((0, \frac{u}{1-u}), (0, \frac{v}{1-v}))\right) \\ &= |\frac{u}{1-u} - \frac{v}{1-v}| - \frac{|u-v|}{1-|u-v|} \geq 0, \end{aligned}$$

since $u + v \geq |u - v| + uv$. Therefore, $\{S, T\}$ is a generalized **Z**-proximal contraction pair. On the other hand, there does not exist $\alpha \in [0, 1[$ such that

$$d((0, u), (0, v)) = |u - v| \leq \alpha \frac{|u-v|}{1-u-v+uv} = \alpha d\left(\frac{u}{1-u}, \frac{v}{1-v}\right) = \alpha d((0, x), (0, y)),$$

for all $u, v \in [0, \frac{1}{3}]$, and hence $\{S, T\}$ is not a generalized proximal contraction pair, it suffices to choose u and v close enough to 0 so that $(1 - u - v + uv)$ approaches 1 and surpasses α, and we conclude that there is no $\alpha \in [0, 1[$ for which the original inequality holds for all pairs (u, v) in that interval.

Remark 2.5. If $\{S, T\}$ is a generalized **Z**-contraction pair and (A, B) has the P-property, then $\{S, T\}$ is a generalized **Z**-proximal contraction pair.

Let A be a nonempty subset of a metric space (X, d). We explain

$$\mathrm{H}_A = \{h : A \to A \text{ is continuous} : d(x, y) \leq d(hx, hy), \ \forall x, y \in A\}.$$

In the following, we prove the existence and uniqueness of a common best proximity point for two non-self mappings without requiring continuity.

Theorem 2.6. *Let A and B be nonempty subsets of a complete metric space (X, d). Moreover, assume that A_0 is nonempty and closed. Let the non-self mappings $S, T : A \to B$ and self-mapping $h : A \to A$ satisfy the following conditions:*

i) $h \in \mathrm{H}_A$;
ii) $\{S, T\}$ *is a generalized* **Z**-*proximal contraction pair;*
iii) $T(A_0), S(A_0) \subseteq B_0$ *and* $A_0 \subseteq h(A_0)$.

Then, there exists a unique point $a \in A$ such that $d(ha, Sa) = d(ha, Ta) = d(A, B)$.

Proof. Fix a_0 in A_0. Since $S(A_0) \subseteq B_0$ and $A_0 \subseteq h(A_0)$, there exists an element a_1 in A_0 such that $d(ha_1, Sa_0) = d(A, B)$. Similarly, since $T(A_0) \subseteq B_0$ and $A_0 \subseteq h(A_0)$, we can choose $a_2 \in A_0$ such that $d(ha_2, Ta_1) = d(A, B)$. Continuing this process, we achieve a sequence $\{a_n\} \subseteq A_0$ such that

$$\begin{cases} d(ha_{2n+1}, Sa_{2n}) = d(A, B), \\ d(ha_{2n+2}, Ta_{2n+1}) = d(A, B). \end{cases}$$

If there exists $m \in \mathbb{N}$ such that $d(a_m, a_{m+1}) = 0$, then $a_n = a_m$ for all $n \geq m$, and thus $d(ha_m, Sa_m) = d(ha_m, Ta_m) = d(A, B)$ (see [19]).
Now, suppose that $0 < d(a_n, a_{n+1}) \leq d(ha_n, ha_{n+1})$ for all $n \in \mathbb{N}$.
First, we prove that

$$\lim_{n\to\infty} d(a_{n-1}, a_n) = 0.$$

Since $h \in \mathrm{H}_A$ and $\{S, T\}$ is a generalized **Z**-proximal contraction pair, we have

$$\begin{aligned} 0 &\leq \zeta(d(ha_{2n+1}, ha_{2n+2}), d(a_{2n}, a_{2n+1})) \\ &< d(a_{2n}, a_{2n+1}) - d(ha_{2n+1}, ha_{2n+2}) \\ &\leq d(a_{2n}, a_{2n+1}) - d(a_{2n+1}, a_{2n+2}), \end{aligned} \tag{2.1}$$

and hence

$$d(a_{2n+1}, a_{2n+2}) \leq d(a_{2n}, a_{2n+1}). \tag{2.2}$$

Similarly,

$$d(a_{2n}, a_{2n+1}) \leq d(a_{2n-1}, a_{2n}). \tag{2.3}$$

From (2.2) and (2.3), it follows that $d(a_n, a_{n+1}) \leq d(a_{n-1}, a_n)$. This implies that the sequence $d(a_{n-1}, a_n)$ is decreasing and so there is a $d \geq 0$ such that $d(a_{n-1}, a_n) \to d$.
Using the property (ζ_1) of a simulation function and (2.1), we conclude that

$$d(ha_{2n+1}, ha_{2n+2}) \leq d(a_{2n}, a_{2n+1}), \qquad \forall n \in \mathbb{N}.$$

From the above inequality and $h \in \mathrm{H}_A$, we have

$$d(a_{2n+1}, a_{2n+2}) \leq d(ha_{2n+1}, ha_{2n+2}) \leq d(a_{2n}, a_{2n+1}), \qquad \forall n \in \mathbb{N}.$$

Therefore, $\lim_{n\to\infty} d(ha_{2n+1}, ha_{2n+2}) = d$.
Suppose that $d > 0$. Using the property (ζ_2) of the simulation function, with $p_n = d(ha_{2n+1}, ha_{2n+2})$ and $q_n = d(a_{2n}, a_{2n+1})$, we obtain

$$0 \leq \limsup_{n\to\infty} \zeta(d(ha_{2n+1}, ha_{2n+2}), d(a_{2n}, a_{2n+1})) < 0,$$

which is a contradiction. Thus, we conclude that $d = 0$.
Next, we show that $\{a_n\}$ is a Cauchy sequence. Since $\lim_{n\to\infty} d(a_{n-1}, a_n) = 0$, it suffices to prove that the subsequence $\{a_{2n}\}$ of $\{a_n\}$ is Cauchy in A_0.
Assume, for contradiction, that there exists an $\epsilon > 0$ for which the subsequences $\{a_{2m(k)}\}$ and $\{a_{2n(k)}\}$ satisfy $n(k) > m(k) > k$ and

$$d(a_{2m(k)}, a_{2n(k)}) \geq \epsilon, \tag{2.4}$$

where $n(k)$ is the smallest integer with this property. Consequently,

$$d(a_{2m(k)}, a_{2n(k)-2}) < \epsilon. \tag{2.5}$$

Using (2.4),(2.5), and the triangle inequality, we derive

$$\begin{aligned}\epsilon \leq d(a_{2m(k)}, a_{2n(k)}) &\leq d(a_{2m(k)}, a_{2n(k)-2}) + d(a_{2n(k)-2}, a_{2n(k)-1}) + d(a_{2n(k)-1}, a_{2n(k)}) \\ &< \epsilon + d(a_{2n(k)-2}, a_{2n(k)-1}) + d(a_{2n(k)-1}, a_{2n(k)}).\end{aligned}$$

Taking the limit as $k \to \infty$ in the above inequality and using $d(a_{n-1}, a_n) \to 0$, we obtain

$$\lim_{k\to\infty} d(a_{2m(k)}, a_{2n(k)}) = \epsilon. \tag{2.6}$$

On the other hand,

$$d(a_{2m(k)}, a_{2n(k)}) \leq d(a_{2m(k)}, a_{2n(k)+1}) + d(a_{2n(k)+1}, a_{2n(k)}),$$

which implies

$$\epsilon \leq \lim_{k\to\infty} d(a_{2m(k)}, a_{2n(k)+1}). \tag{2.7}$$

Similarly,

$$d(a_{2m(k)}, a_{2n(k)+1}) \leq d(a_{2m(k)}, a_{2n(k)}) + d(a_{2n(k)}, a_{2n(k)+1}).$$

Taking the limit as $k \to \infty$ in the above inequality and applying (2.6), we get

$$\lim_{k\to\infty} d(a_{2m(k)}, a_{2n(k)+1}) \leq \epsilon. \tag{2.8}$$

From (2.7) and (2.8), we conclude

$$\lim_{k\to\infty} d(a_{2m(k)}, a_{2n(k)+1}) = \epsilon \tag{2.9}$$

Analogously, we can show

$$\lim_{k\to\infty} d(a_{2m(k)+1}, a_{2n(k)+2}) = \epsilon. \tag{2.10}$$

We may assume $d(a_{2m(k)}, a_{2n(k)+1}) > 0$ and $d(a_{2m(k)+1}, a_{2n(k)+2}) > 0$ for all $k \in \mathbb{N}$. Since (S, T) is a generalized **Z**-proximal contraction pair and

$$\begin{cases} d(ha_{2m(k)+1}, Sa_{2m(k)}) = d(A, B), \\ d(ha_{2n(k)+2}, Ta_{2n(k)+1}) = d(A, B), \end{cases} \tag{2.11}$$

we obtain

$$\begin{aligned} 0 &\leq \zeta(d(ha_{2m(k)+1}, ha_{2n(k)+2}), d(a_{2m(k)}, a_{2n(k)+1})) \\ &< d(a_{2m(k)}, a_{2n(k)+1}) - d(ha_{2m(k)+1}, ha_{2n(k)+2}).\end{aligned}$$

By the above inequality and $h \in \mathrm{H}_A$, we have

$$d(a_{2m(k)+1}, a_{2n(k)+2}) \leq d(ha_{2m(k)+1}, ha_{2n(k)+2}) \leq d(a_{2m(k)}, a_{2n(k)+1}).$$

Letting $k \to \infty$ in the above inequality and using (2.9) and (2.10), we obtain

$$\lim_{k\to\infty} d(ha_{2m(k)+1}, ha_{2n(k)+2}) = \epsilon. \tag{2.12}$$

By using the property (ζ_2) of simulation function, (2.9) and (2.12), with $p_k = d(ha_{2m(k)+1}, ha_{2n(k)+2})$ and $q_k = d(a_{2m(k)}, a_{2n(k)+1})$, we obtain

$$0 \leq \limsup_{k\to\infty} \zeta(d(ha_{2m(k)+1}, ha_{2n(k)+2}), d(a_{2m(k)}, a_{2n(k)+1})) < 0,$$

which is a contradiction. Thus, $\{a_n\}$ is a Cauchy sequence.
Since (X, d) is complete and A_0 is closed, then A_0 is complete and hence there exists $a \in A_0$ such that $a_n \to a$. Moreover, by the continuity of h, we have $ha_n \to ha$, and thus $ha \in A_0$, since $ha_n \in A_0$ for all $n \in \mathbb{N}$ and A_0 is closed.
On the other hand, since $a \in A_0$ and $T(A_0) \subset B_0$, there exists $u \in A_0$ such that

$$d(u, Ta) = d(A, B). \tag{2.13}$$

Now, if $u = ha_n$ for infinitely many $n \in \mathbb{N}$, then $u = ha$. Otherwise, assume $u \neq ha_n$ for all $n \in \mathbb{N}$. Additionally, there exists a subsequence $\{a_{2n(k)}\}$ of $\{a_n\}$ such that $a_{2n(k)} \neq a$, for all $k \in \mathbb{N}$.
By (2.11), (2.13), and since $\{S, T\}$ is a generalized **Z**-proximal contraction pair, we have

$$0 \leq \zeta(d(u, ha_{2n(k)+1}), d(a, a_{2n(k)})) < d(a, a_{2n(k)}) - d(u, ha_{2n(k)+1}),$$

therefore

$$d(u, ha_{2n(k)+1}) < d(a, a_{2n(k)}), \quad \forall k \in \mathbb{N}.$$

If $k \to \infty$, we obtain $d(u, ha_{2n(k)+1}) \to 0$, and hence $ha = u$. Thus, we have proved that $d(ha, Ta) = d(A, B)$. Similarly, we can prove $d(ha, Sa) = d(A, B)$.
Therefore,

$$d(ha, Ta) = d(ha, Sa) = d(A, B).$$

To prove the uniqueness, let $b \neq a$ be another point in A_0 such that

$$d(hb, Sb) = d(hb, Tb) = d(A, B).$$

Since $h \in \mathrm{H}_A$ and $\{S, T\}$ is a generalized **Z**-proximal contraction pair, we have

$$\begin{aligned} 0 &\leq \zeta(d(ha, hb), d(a, b)) \\ &< d(a, b) - d(ha, hb) \\ &\leq d(a, b) - d(a, b) = 0, \end{aligned}$$

which is a contradiction.

Let us illustrate the above theorem with the following examples.

Example 2.7. Consider $X = \mathbb{R}$ with the usual metric $d(x, y) = |x - y|$. Let $A = \{-6, 0, 6\}$, $B = \{-4, -1, 4\}$ and $\zeta = \zeta_\lambda$ be as in Example 1.5. Then, A and B are nonempty subsets of X with $d(A, B) = 1$, $A_0 = \{0\}$ and $B_0 = \{-1\}$.
We define the mappings $S, T : A \to B$ by:

$$S(-6) = -4, \quad S(0) = -1, \quad S(6) = 4, \quad \textit{and} \quad T(a) = -1 \quad \forall a \in A.$$

It is immediate to see that $S(A_0), T(A_0) \subseteq B_0$. Also, if

$$\begin{cases} d(u, Sx) = d(A, B) = 1 \\ d(v, Ty) = d(A, B) = 1, \end{cases}$$

then $u = v = x = 0$ and $y \in A$. Therefore, $\{S, T\}$ is a generalized **Z**-proximal contraction pair. Let $h : A \to A$ be defined as $h(a) = a$. Then, it can be observed that $A_0 \subseteq h(A_0)$ and $h \in \mathrm{H}_A$. Hence, all the conditions of Theorem 2.6 hold for this example, and clearly, 0 is the common best proximity point of S and T

The following example shows that the condition $A_0 \subseteq h(A_0)$ is necessary and important, and without it, the results of Theorem 2.6 may not hold.

Example 2.8. Let $X = \mathbb{R}$ be equipped with the Euclidean metric $d(x, y) = |x - y|$ for all $x, y \in X$, and let $A = \{1, 2\}$ and $B = \{3, 4\}$. Define the operators $S, T : A \to B$ as

$$S(1) = 4, \quad S(2) = 3, \quad T(1) = 4, \quad T(2) = 3.$$

It is obvious that $d(A, B) = 1$, $A_0 = \{2\}$ and $B_0 = \{3\}$. Also,
$S(A_0) = S(\{2\}) = \{3\} \subseteq B_0 = \{3\}$ and $T(A_0) = T(\{2\}) = \{3\} \subseteq B_0$.
It is easy to show that $\{S, T\}$ is a generalized **Z**-proximal contraction pair, where the function $\zeta = \zeta_\lambda$ is given in Example 1.5. If

$$\begin{cases} d(u, Sx) = d(A, B) = 1, \\ d(v, Ty) = d(A, B) = 1, \end{cases}$$

we get $u = v = x = y = 2$, and thus $\zeta_\lambda(d(u, v), d(x, y)) \geq 0$.
Now, suppose $h(1) = 2$ and $h(2) = 1$. Then $h \in \mathrm{H}_A$ but $A_0 \nsubseteq h(A_0)$.
Next, we show that the non-self mappings S and T do not have a common best proximity point. Since
If $a = 1$, then $d(h(1), S(1)) = d(2, 4) = 2 \neq d(A, B)$.
If $a = 2$, then $d(h(2), S(2)) = d(1, 3) = 2 \neq d(A, B)$.
Therefore, all the conditions of Theorem 2.6 except $A_0 \subseteq h(A_0)$ hold, and thus S and T do not have a common best proximity point.

The following corollary is immediate consequence of above theorem by setting h as the identity mapping on A.

Corollary 2.9. *Let A and B be nonempty subsets of a complete metric space (X, d). Moreover, assume that A_0 is nonempty and closed. Let also the non-self mappings $S, T : A \to B$ satisfy the following conditions:*

i) *$\{S, T\}$ is a generalized* **Z***-proximal contraction pair;*
ii) *$T(A_0), S(A_0) \subseteq B_0$.*

Then the functions S and T have a unique common best proximity point.

From Theorem 2.6 with $S = T$, we obtain the following corollary, which is proved in the Theorem 3.1 of [19].

Corollary 2.10. *Let A and B be nonempty subsets of a complete metric space (X, d). Moreover, assume that A_0 is nonempty and closed. Let also that the mappings $S : A \to B$ and $h : A \to A$ satisfy the following conditions:*

i) *$h \in \mathrm{H}_A$;*
ii) *S is a* **Z***-proximal contraction of the first kind;*
iii) *$S(A_0) \subseteq B_0$ and $A_0 \subseteq h(A_0)$.*

Then there exists a unique point $a \in A$ such that $d(ha, Sa) = d(A, B)$. Moreover, for every $a_0 \in A_0$ there exists a sequence $\{a_n\} \subset A$ such that $d(ha_{n+1}, Sa_n) = d(A, B)$, for all $n \in \mathbb{N} \cup \{0\}$ and $a_n \to a$.

We propose two common best proximity point theorems for generalized proximal contractions and generalized **Z**-contractions, which can be considered as extensions of Theorem 2.6.

Theorem 2.11. *Let A and B be nonempty subsets of a complete metric space (X, d). Moreover, assume that A_0 is nonempty and closed. Let also that the mappings $S, T : A \to B$, and $h : A \to A$ satisfy the following conditions:*

i) $h \in \mathrm{H}_A$;
ii) $\{S, T\}$ *is a generalized proximal contraction pair;*
iii) $S(A_0), T(A_0) \subseteq B_0$ *and* $A_0 \subseteq h(A_0)$.

Then there exists a unique point $a \in A$ such that $d(ha, Sa) = d(ha, Ta) = d(A, B)$.

Proof. If $\{S, T\}$ is a generalized proximal contraction pair, then $\{S, T\}$ is a generalized **Z**-proximal contraction pair concerning the simulation function $\zeta_\lambda : [0, \infty[\times[0, \infty[\to \mathbb{R}$ defined by $\zeta_\lambda(p, q) = \lambda q - p$, for all $p, q \in [0, \infty[$ and $\lambda \in [0, 1[$.

Theorem 2.12. *Let A and B be nonempty subsets of a complete metric space (X, d). Moreover, assume that A_0 is nonempty and closed. Let also that the mappings $S, T : A \to B$, and $h : A \to A$ satisfy the following conditions:*

i) $h \in \mathrm{H}_A$;
ii) *pair (A, B) has the P-property;*
iii) $\{S, T\}$ *is a generalized* **Z***-contraction pair;*
iv) $S(A_0), T(A_0) \subseteq B_0$ *and* $A_0 \subseteq h(A_0)$.

Then there exists a unique point $a \in A$ such that $d(ha, Sa) = d(ha, Ta) = d(A, B)$.

Proof. If $\{S, T\}$ is a generalized **Z**-contraction pair and (A, B) has the P-property then $\{S, T\}$ is a generalized **Z**-proximal contraction pair.
Let us illustrate the above theorem with the following examples.

Example 2.13. Consider $X = \mathbb{R}$ with the usual metric $d(x, y) = |x - y|$. Let $A = \{0\} \cup [1, 2]$, $B = \{3\} \cup [4, 5]$ and $\zeta(p, q) = \frac{1}{2}q - p$. Then, A and B are nonempty subsets of X with $d(A, B) = 1$, $A_0 = \{2\}$ and $B_0 = \{3\}$. Define the mappings $S, T : A \to B$ by $S(x) = 3$ and $T(x) = 3$. Then, $\{S, T\}$ is a generalized **Z**-contraction pair such that $S(A_0), T(A_0) \subseteq B_0$. Let $h : A \to A$ be defined as $h(a) = a$. Then, it can be observed that $A_0 \subseteq h(A_0)$ and $h \in \mathrm{H}_A$. Clearly, (A, B) has the P-property and then all the conditions of Theorem 2.12 hold for this example, and clearly, 2 is the common best proximity point of S and T.

The following example shows that the condition P-property in Theorem 2.12 can not be relaxed to ensure the existence of a best proximity point for a non-self generalized **Z**-contraction mapping.

Example 2.14. Fix $\mathbf{R} > 0$ and $r = \frac{\mathbf{R}}{4}$. Consider $X = \mathbb{C}$ with the Euclidean metric, $A = \{\mathbf{R}e^{i\phi} : 0 \leq \phi \leq 2\pi\}$ and $B = \{re^{i\phi} : 0 \leq \phi \leq 2\pi\}$. Then A, B are nonempty closed subsets of X with $A_0 = A$, $B_0 = B$ and $d(A, B) = \frac{3}{4}R$. Clearly, (A, B) does not have a P-property, because

$$d(\mathbf{R}, r) = d(\mathbf{R}e^{i\frac{\pi}{2}}, re^{i\frac{\pi}{2}}) = d(A, B) \nRightarrow d(\mathbf{R}, \mathbf{R}e^{i\frac{\pi}{2}}) = d(r, re^{i\frac{\pi}{2}}).$$

Consider the operators $S, T : A \to B$ by $S(\mathbf{R}e^{i\phi}) = T(\mathbf{R}e^{i\phi}) = re^{i(\phi+\pi)}$, for all $\phi \in [0, 2\pi]$ and $h(a) = a$, for all $a \in A$. Then $\{S, T\}$ is a generalized $\mathbf{Z}$-contraction pair with respect to the simulation function $\zeta : [0, \infty[\times[0, \infty[\to \mathbb{R}$ defined by $\zeta(p, q) = \lambda q - p$, for all $p, q \in [0, \infty[$ and $\lambda \in [\frac{1}{4}, 1[$ with $T(A_0), S(A_0) \subseteq B_0$. Clearly, the pair $\{S, T\}$ does not have a common best proximity point.

3. Variational inequality problems

Let $\mathcal{H}$ be a real Hilbert space with inner product $\langle ., . \rangle$ and induced norm $\| \; . \; \|$. Let $\{M_j\}_{j=1}^K$ be nonempty, closed and convex subsets of $\mathcal{H}$ with $M = \bigcap_{j=1}^K M_j \neq \emptyset$. We discuss a CSVIP (common solutions to variational inequality problem) as follows; see [3, 4, 5, 15].

problem 3.1. Find $a^\star \in M = \bigcap_{j=1}^K M_j \neq \emptyset$ such that

$$\langle F_j(a^\star), a - a^\star \rangle \geq 0 \quad forall \quad a \in M_j, j = 1, 2, ..., K, \tag{3.1}$$

where $\{F_j\}_{j=1}^K : M_j \to \mathcal{H}$ are given monotone operators (*i.e.*, $\langle F_j(a) - F_j(b), a - b \rangle \geq 0$ for all $a, b \in M_j, j = 1, 2, ..., K$).

If $K = 1$, the CSVIP (3.1) reduces to the VIP (classical variational inequality problem)[8]. In this part of the research, we assume $K = 2$. Denoting $F_1 = f$, $F_2 = g$ and the nonempty, closed and convex subsets M_1 and M_2 by C and Q, respectively, we obtain the following two-set CSVIP:

problem 3.2. Find $a^\star \in M_1 \bigcap M_2 \neq \emptyset$ such that

$$\langle f(a^\star), a - a^* \rangle \geq 0 \qquad \text{for all } a \in M_1,$$

and

$$\langle g(a^\star), a - a^* \rangle \geq 0 \qquad \text{for all } a \in M_2.$$

A wide class of equilibrium problems arising in pure and applied sciences can be solved using variational inequality theory [16]. We recall the metric projection $\mathcal{P}_M : \mathcal{H} \to M$, a crucial tool for analyzing VIP. It is known that for every $a \in \mathcal{H}$, there exists a unique nearest point $\mathcal{P}_M(a) \in M$ satisfying the condition

$$\| a - \mathcal{P}_M(a) \| \leq \| a - b \|, \qquad \text{for all } b \in M.$$

Next, we state the basic lemmas that play an essential role in establishing the existence of a common solution for a variational inequality problem and the existence of a common fixed point for certain mappings.

Lemma 3.3. *Let $c \in \mathcal{H}$. Then, $a \in M$ satisfies the inequality*

$$\langle a - c, d - a \rangle \geq 0 \quad \textit{for all} \quad d \in M$$

if and only if $a = \mathcal{P}_M(c)$.

Lemma 3.4. *Let $f : \mathcal{H} \to \mathcal{H}$ be monotone. Then, $a \in M$ is a solution of*

$$\langle f(a), b - a \rangle \geq 0 \quad \textit{for all} \quad b \in M$$

if and only if $a = \mathcal{P}_M(a - \alpha f a)$*, where* $\alpha > 0$*.*

Theorem 3.5. *Let* M *be nonempty, closed and convex subsets of a real Hilbert space* $\mathcal{H}$*, and let monotone operators* $f, g : \mathcal{H} \to \mathcal{H}$ *be such that* $\mathcal{P}_M(I - \alpha f), \mathcal{P}_M(I - \alpha g) : M \to M$ *for some* $\alpha > 0$*. In addition, suppose that* $\{\mathcal{P}_M(I-\alpha f), \mathcal{P}_M(I-\alpha g)\}$ *is a generalized* **Z***-proximal contraction pair. Then, there exists a unique point* $a \in M$ *such that*

$$\langle f(a), b - a\rangle \geq 0 \quad \textit{and} \quad \langle g(a), b - a\rangle \geq 0 \quad \textit{for all} \quad b \in M.$$

Proof. We define the operators $S, T : M \to M$ by

$$Sx = \mathcal{P}_M(x - \alpha f(x)) \quad \text{and} \quad Tx = \mathcal{P}_M(x - \alpha g(x)).$$

By lemma 3.4, $a \in M$ is a common solution of

$$\langle f(a), b - a\rangle \geq 0 \quad \text{and} \quad \langle g(a), b - a\rangle \geq 0 \quad \text{for all} \quad b \in M,$$

or equivalently, a solution of problem 3.2 with $M_1 = M_2 = M$, if and only if $a = Sa = Ta$. The pair $\{S, T\}$ satisfies all the conditions of the Theorem 2.6 by putting $A = B = M$ and $h = I$. Therefore, Theorem 3.5 follows as a direct consequence of Theorem 2.6.

Example 3.6. Let our Hilbert space be $\mathcal{H} = \mathbb{R}^2$, equipped with the standard inner product defined by $\langle x, y\rangle = x_1y_1 + x_2y_2$ and the corresponding norm $\|x\| = \sqrt{\langle x, x\rangle}$.
Consider the set $M = \{x = (x_1, x_2) \in \mathbb{R}^2 \mid \|x\| \leq 1\}$, which is the closed unit disk. This set is closed, convex, bounded, and non-empty.
We define two monotone operators $f, g : H \to H$ by $f(x) = x$ and $g(x) = x$. Note that these are monotone because $\langle f(x) - f(y), x - y\rangle = \|x - y\|^2 \geq 0$, and similarly for g.
Next, we define the mappings $S, T : H \to M$ as follows:

$$S(x) = \mathcal{P}_M(x - \alpha f(x)) = \mathcal{P}_M((1 - \alpha)x)$$

$$T(x) = \mathcal{P}_M(x - \alpha g(x)) = \mathcal{P}_M((1 - \alpha)x)$$

where $\mathcal{P}_M$ is the metric projection onto the set M, and $\alpha \in (0, 1)$ is a constant.
For any $x \in M$, since M is convex and contains the origin, the point $(1 - \alpha)x$ also belongs to M. Therefore, the projection onto M is the identity, yielding:

$$S(x) = (1 - \alpha)x \quad \text{and} \quad T(x) = (1 - \alpha)x$$

We now verify whether the pair $\{S, T\}$ is a generalized **Z**-proximal contraction pair with respect to the simulation function $\zeta_\lambda : [0, \infty) \times [0, \infty) \to \mathbb{R}$ defined by:

$$\zeta_\lambda(p, q) = \lambda q - p, \quad \text{for } \lambda \in (0, 1).$$

For any $x, y \in H$, we have:

$$\begin{aligned} \zeta_\lambda(\|S(x) - T(y)\|, \|x - y\|) &= \lambda\|x - y\| - \|(1 - \alpha)x - (1 - \alpha)y\| \\ &= \lambda\|x - y\| - (1 - \alpha)\|x - y\| = \|x - y\|(\lambda - (1 - \alpha)). \end{aligned}$$

If $\lambda > 1 - \alpha$, then $\lambda - (1 - \alpha) > 0$, and the value of ζ_λ is non-negative. Hence, the condition for a generalized **Z**-proximal contraction pair is satisfied.
Therefore, all assumptions of Theorem 3.5 are satisfied. We conclude that there exists a unique point $a \in M$ such that:

$$\langle f(a), b - a\rangle \geq 0 \quad \text{and} \quad \langle g(a), b - a\rangle \geq 0 \quad \text{for all } b \in M.$$

This point is the projection of the origin onto M. Given that M is the unit disk and the origin is already inside M, the projection is $a = 0$. Substituting $a = 0$ into the inequalities confirms the result, as $\langle f(0), b\rangle = \langle 0, b\rangle = 0 \geq 0$.

4. Conclusions

Fixed point theorems provide solutions to equations of the form $Sa = a$, where S is self mapping. However, if S is a non-self mapping, there is no guarantee for the existence of a solution. In such cases, best proximity point theorems offer approximate solutions to nonlinear problems. The literature contains numerous works addressing the existence of best proximity points for various types of non-self mappings. A more general extension of these theorems, involving multiple non-self mappings, is known as common best proximity point theorems which have been extensively investigated by researchers. In this work, we introduce new concepts, including generalized **Z**-contraction pairs, generalized proximal contraction pairs, and generalized **Z**- proximal contraction pairs for non-self mappings, using the notion of a simulation function. We establish existence and uniqueness theorems for a common best proximity point in a complete metric space. Additionally, we provide several illustrative examples to validate our main result. Finally, we discuss the solvability of common solutions to variational inequality problems in Hilbert spaces, further supporting our key findings.

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