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DUAL OF THE DUPLICATION OF A MODULE ALONG AN IDEAL

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ABSTRACT

Let R be a commutative ring with identity, I be an ideal of R, and M be an R-module. The amalgamated duplication of R along I, denoted by $R \bowtie I$, is the special subring of $R \times R$ defined by $R \bowtie I := \{(a, a+i) | a \in R, i \in I\}$. In this paper, we introduce and investigate a special kind of $R \bowtie I$-modules, called the dual of the duplication of M along I defined by $M \bowtie^d I = \{(m_1, m_2) \in M \times M | m_1 - m_2 \in (0 :_M Ann_R(I))\}$. In fact the $R \bowtie I$-module $M \bowtie^d I$ is a dual of the $R \bowtie I$-module $M \bowtie I := \{(m, m') \in M \times M | m - m' \in IM\}$ which is introduced and studied by E. M. Bouba, N. Mahdou and M. Tamekkante.

1. Introduction

Throughout this paper, R will denote a commutative ring with identity. Let I be an ideal of R. The *amalgamated duplication of* R *along* I, denoted by $R \bowtie I$, is the special subring of $R \times R$ defined by $R \bowtie I := \{(a, a+i) | a \in R, i \in I\}$. Clearly, R is embedded in $R \bowtie I$ by the mapping $a \mapsto (a, a)$, and this new ring can be thought of as an extension of R. In [3], the authors studied on the structure of the amalgamated algebra with modules. Let I be an ideal of R and M be an R-module. The *duplication of the* R*-module* M *along the ideal* I denoted by $M \bowtie I$ and defined by

$$M \bowtie I := \{(m, m') \in M \times M | m - m' \in IM\}$$

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which is an $R \bowtie I$-module with the multiplication given by $(a, a+i).(m, m') = (am, (a+i)m')$, where $a \in R, i \in I$, and $(m, m') \in M \bowtie I$ [3]. Clearly, if $M = R$, then the duplication of the R-module R along the ideal I coincides with the amalgamated duplication of the ring R along the ideal I.

Let I be an ideal of R and M be an R-module. The purpose of this paper is to introduce and investigate a special kind of $R \bowtie I$-modules, called the *dual of the duplication of M along I* defined by

$$M \bowtie^d I = \{(m_1, m_2) \in M \times M | m_1 - m_2 \in (0 :_M Ann_R(I))\}.$$

In fact the $R \bowtie I$-module $M \bowtie^d I$ is a dual of the $R \bowtie I$-module $M \bowtie I$.

2. Main Results

Definition 2.1. Let I be an ideal of R and M be an R-module. We define the *dual of the duplication of the R-module M along the ideal I* by $M \bowtie^d I = \{(m_1, m_2) \in M \times M | m_1 - m_2 \in (0 :_M Ann_R(I))\}$ which is an $R \bowtie I$-module with the multiplication given by $(a, a + i).(m, m') = (am, (a + i)m')$, where $a \in R, i \in I$, and $(m, m') \in M \bowtie^d I$. In fact, $M \bowtie^d I$ is a dual of $M \bowtie I$.

Example 2.2. Let $R = \mathbb{Z}_{12}$, $M = \mathbb{Z}_{12}$, and $I = 3\mathbb{Z}_{12}$. Then

$$M \bowtie I = M \bowtie^d I = \{(3, 6), (6, 3), (1, 4), (4, 1), (5, 2), (2, 5), (4, 7), (7, 4), (8, 5), (5, 8), (9, 6), (6, 9), (10, 7), (7, 10), (11, 8), (8, 11), (0, 12), (12, 0)\}.$$

Recall that an R-module M is said to be a *multiplication module* if for every submodule N of M there exists an ideal I of R such that $N = IM$ [2]. It is easy to see that M is a multiplication module if and only if $N = (N :_R M)M$ for each submodule N of M. Also, an R-module M is said to be a *comultiplication module* if for every submodule N of M there exists an ideal I of R such that $N = (0 :_M I)$ [1]. It is easy to see that M is a comultiplication module if and only if $N = (0 :_M Ann_R(N))$ for each submodule N of M.

In the following proposition, for an ideal I of R and R-module M we investigate the relationships between $M \bowtie I$ and $M \bowtie^d I$.

Proposition 2.3. Let I be an ideal of R and M be an R-module. Then we have the following.

(a) $M \bowtie I \subseteq M \bowtie^d I$.
(b) If M is a faithful comultiplication R-module, then $M \bowtie^d I = M \bowtie I$.
(c) If M is a faithful multiplication R-module, then $M \bowtie^d Ann_R(I) = M \bowtie Ann_R(I)$.
(d) If R is a comultiplication R-module and M is a faithful multiplication R-module, then $M \bowtie^d I = M \bowtie I$.
(e) If $I \neq 0$ and $Ann_R(I) \subseteq Ann_R(M)$, then $M \bowtie^d I = M \times M$. In particular, if R is an integral domain, then $M \bowtie^d I = M \times M$.

Proof. (a) $Ann_R(I)IM = 0$ implies that $Ann_R(I) \subseteq Ann_R(IM)$. Clearly, $IM \subseteq (0 :_M Ann_R(IM))$. Therefore, $IM \subseteq (0 :_M Ann_R(I))$, as needed.

(b) Let M be a faithful comultiplication R-module. Since M is faithful, $Ann_R(IM) = Ann_R(I)$. Now the result follows from the fact that $IM = (0 :_M Ann_R(IM))$ because M is a comultiplication module.

(c) Assume that M is a faithful multiplication R-module. Then

$$(0:_M I) = ((0:_M I):_R M)M = Ann(IM)M = Ann_R(I)M.$$

Now since $(0:_M Ann_R(Ann_R(I))) \subseteq (0:_M I)$, we have $M \bowtie^d Ann_R(I) \subseteq M \bowtie Ann_R(I)$. The reverse inclusion follows from part (a).

(d) Let R be a comultiplication R-module and M be a faithful multiplication R-module. As R is a comultiplication R-module, $I = Ann_R(Ann_R(I))$. Now the result follows from part (c).

(e) This is clear. □

The following example shows that the condition "M is a comultiplication R-module" in the Proposition 2.3 (b) is necessary.

Example 2.4. Let $R = \mathbb{Z}$, $M = \mathbb{Z}$, and $I = n\mathbb{Z} \neq 0$. Then $M \bowtie I = \{(x, y) : x - y \in n\mathbb{Z}\}$ and $M \bowtie^d I = \mathbb{Z} \times \mathbb{Z}$.

Notation 2.5. Let I be an ideal of R and N be a submodule of an R-module M. Set $N \bowtie^d I = \{(m_1, m_2) \in N \times M | m_1 - m_2 \in (0:_M Ann_R(I))\}$. Clearly, $N \bowtie^d I$ is a submodule of $M \bowtie^d I$.

Lemma 2.6. Let I be an ideal of R and N be a submodule of an R-module M. Then we have the following.

(a) $Ann_{R\bowtie I}(N \bowtie^d I) \subseteq Ann_R(N) \bowtie I$.
(b) $(N \bowtie^d I :_{R\bowtie I} M \bowtie^d I) = (N :_R M) \bowtie I$.

Proof. (a) Let $(a, a+i) \in Ann_{R\bowtie I}(N \bowtie^d I)$. Then $(a, a+i)(n, m) = 0$ for each $(n, m) \in N \times M$ with $n - m \in (0:_M Ann_R(I))$. This implies that $(a, a+i)(n, n) = 0$ for each $n \in N$. It follows that $a \in Ann_R(N)$. Therefore, $Ann_{R\bowtie I}(N \bowtie^d I) \subseteq Ann_R(N) \bowtie I$.

(b) Let $(a, a+i) \in (N \bowtie^d I :_{R\bowtie I} M \bowtie^d I)$ and $x \in M$. Then $(x, x) \in M \bowtie^d I$. Thus $(ax, (a+i)x) = (a, a+i)(x, x) \in N \bowtie^d I$. Thus $ax \in N$ and so $a \in (N :_R M)$. Therefore, $(a, a+i) \in (N :_R M) \bowtie I$. Conversely, let $(a, a+i) \in (N :_R M) \bowtie I$ and $(x, y) \in M \bowtie^d I$. Then $aM \subseteq N$ and $(x - y)Ann_R(I) = 0$. This implies that

$$(ax - (a+i)y)Ann_R(I) = ((x - y)a - iy)Ann_R(I) \subseteq$$

$$(x - y)aAnn_R(I) - iyAnn_R(I) = (x - y)Ann_R(I)a - iAnn_R(I)y = 0.$$

Therefore, $(a, a+i)(x, y) = (ax, (a+i)y) \in N \bowtie^d I$. Thus $(a, a+i) \in (N \bowtie^d I :_{R\bowtie I} M \bowtie^d I)$, as needed. □

A non-zero submodule N of an R-module M is called *second* if for each $a \in R$, we have $aN = N$ or $aN = 0$ [12]. For more information about this class of modules we refer the reader to [5]. The dual notion of $Z_R(M)$, the set of zero divisors of M [11], is denoted by $W_R(M)$ and defined by

$$W_R(M) = \{a \in R : aM \neq M\}.$$

Proposition 2.7. Let I be an ideal of R and N be a non-zero submodule of an R-module M. Then we have the following.

(a) If $N \bowtie^d I$ is a second submodule of the $R \bowtie I$-module $M \bowtie^d I$, then N is a second submodule of M.

(b) If M is a second R-module, $Z_R(M) \subseteq W_R(M)$, and $IM = 0$, then $M \bowtie^d I$ is a second submodule of the $R \bowtie I$-module $M \bowtie^d I$

Proof. (a) Let $N \bowtie^d I$ be a second submodule of the $R \bowtie I$-module $M \bowtie^d I$. Let $a \in R$. Then $(a, a) \in R \bowtie I$. By assumption, $(a, a)N \bowtie^d I = 0$ or $(a, a)N \bowtie^d I = N \bowtie^d I$. Let $n \in N$. Then $(n, n) \in N \bowtie^d I$. If $(a, a)N \bowtie^d I = 0$, then $(an, an) = (a, a)(n, n) = 0$. Thus $an = 0$ and hence $aN = 0$. So, suppose that $(a, a)N \bowtie^d I = N \bowtie^d I$. Hence, $(n, n) = (a, a)(n_1, m_1)$ for some $(n_1, m_1) \in N \bowtie^d I$. It follows that $n = an_1 \in aN$. So, $aN = N$. Therefore, N is a second submodule of M.

(b) Let M be a second module and $IM = 0$. Suppose that $(a, a + i) \in R \bowtie I$. By assumption, $aM = 0$ or $aM = M$. If $aM = 0$, then $(a, a + i)M \bowtie^d I = 0$ and we are done. So, suppose that $aM = M$ and $(x, y) \in M \bowtie^d I$. Then $x = am_1$ and $y = am_2$ for some $m_1, m_2 \in M$. Thus by using $IM = 0$, we have

$$(x, y) = (am_1, am_2) = (am_1, am_2 + 0) = (am_1, am_2 + im_2) = (a, a + i)(m_1, m_2).$$

Since $Z_R(M) \subseteq W_R(M)$ and $a \notin W_R(M)$, we get that $(m_1, m_2) \in M \bowtie^d I$. Thus $M \bowtie^d I \subseteq (a, a + i)M \bowtie^d I$ and we are done because the inverse inclusion is clear. □

A proper submodule P of an R-module M is said to be *prime* if for any $r \in R$ and $m \in M$ with $rm \in P$, we have $m \in P$ or $r \in (P :_R M)$ [6, 4]. For more information about some generalizations of this class of modules we refer the reader to [10].

Lemma 2.8. Let N be a submodule of an R-module M. Then the following statements are equivalent:

(a) N is a prime submodule of the R-module M;

(b) $N \bowtie^d I$ is a prime submodule of the $R \bowtie I$-module $M \bowtie^d I$.

Proof. By using Lemma 2.6 (b), the proof is similar to the proof of [7, Lemma 11]. □

Example 2.9. Let p be a prime number and $n > 1$. Since $p\mathbb{Z}$ is a prime submodule of the $\mathbb{Z}$-module $\mathbb{Z}$. We have $p\mathbb{Z} \bowtie^d n\mathbb{Z}$ is a prime submodule of $\mathbb{Z} \bowtie^d n\mathbb{Z}$ by Lemma 2.8.

Let N be a submodule of an R-module M. By M-$rad(N)$, we denote the M-radical of N which is the intersection of all prime submodules containing the submodule N, and M-$rad(N) = M$ in case N is not contained in any prime submodule of M.

Theorem 2.10. *Let N be a submodule of an R-module M. Then the prime submodule which contains $N \bowtie^d I$ are exactly of the form $L \bowtie^d I$ where L is a prime submodule of M containing N. Moreover, $M \bowtie^d I$-$rad(N \bowtie^d I) = M$-$rad(N) \bowtie^d I$.*

Proof. Let $\varphi : M \bowtie^d I \longrightarrow M; (m, m') \longmapsto m$ be an $R \bowtie I$-epimorphism. Then $\ker \varphi = 0 \times (0 :_M Ann_R(I))$. Therefore, $M \bowtie^d I/0 \times (0 :_M Ann_R(I)) \simeq M$. Thus if L is a submodule of $M \bowtie^d I$ such that $0 \times (0 :_M Ann_R(I)) \subset L$, then there exists a submodule K of M such that $L/0 \times (0 :_M Ann_R(I)) \simeq K$, which implies that $L = K \bowtie^d I$. By using Lemma 2.8, we have that L is a prime submodule of $M \bowtie^d I$ if and only if K is a prime submodule of M. Now, if L is a prime submodule of $M \bowtie^d I$ such that $N \bowtie^d I \subset L$, then $0 \times (0 :_M Ann_R(I)) \subset L$, and so there exits a prime submodule K of M with $L = K \bowtie^d I$. Now, it is clear that $\bigcap_i (K_i \bowtie^d I) = (\bigcap_i K_i) \bowtie^d I$ where K_i's are the submodules of M and so we have $M \bowtie^d I$-$rad(N \bowtie^d I) = M$-$rad(N) \bowtie^d I$. □

Let S be a multiplicatively closed subset of R and M be an R-module. A submodule N of M with $(N :_R M) \cap S = \emptyset$ is said to be an S-1–*absorbing primary* if there exists an (fixed) $s \in S$ such that for all non-unit $a, b \in R$ and $m \in M$ if $abm \in N$, then either $sab \in (N :_R M)$ or $sm \in M\text{-}rad(N)$. This fixed element $s \in S$ is called an S-*element* of N [7].

Theorem 2.11. *Let S be a multiplicatively closed subset of R, I be an ideal of R, and M be an R-module. Let N be a submodule of M with $(N :_R M) \cap S = \emptyset$. Then the following statements are equivalent:*

(a) *N is an S-1-absorbing primary submodule of M.*
(b) *$N \bowtie^d I$ is an $S \bowtie I$-1-absorbing primary submodule of $M \bowtie^d I$.*

Proof. By using Lemma 2.6 (b), we have $(N \bowtie^d I :_{R\bowtie I} M \bowtie^d I) \cap S \bowtie I = \emptyset$ if and only if $(N :_R M) \cap S = \emptyset$.

$(a) \Rightarrow (b)$ Let $s \in S$ be an S-element of N and let $(a, a+i)(b, b+i)(m, m') \in N \bowtie^d I$ for non-unit elements $(a, a+i), (b, b+i) \in R \bowtie I$ and $(m, m') \in M \bowtie^d I$. Then $abm \in N$. If a or b is unit, the claim is clear. Now we can assume a and b are non-unit elements of R. Due to the fact that N is an S-1-absorbing primary submodule of R, we get either $sab \in (N :_R M)$ or $sm \in M\text{-}rad(N)$. In the first case, we conclude that $(s, s)(a, a+i)(b, b+i) \in (N :_R M) \bowtie I = (N \bowtie^d I :_{R\bowtie I} M \bowtie^d I)$. In the second case, we have $(sm, sm' + im') \in M\text{-}rad(N) \bowtie^d I$, since $r(m - m') - im' \in (0 :_M I)$. Thus, the claim follows from $M\text{-}rad(N) \bowtie^d I = M \bowtie^d I\text{-}rad(N \bowtie^d I)$ by Theorem 2.10.

$(b) \Rightarrow (a)$ Let $(s, s+i)$ be an $S \bowtie I$-element of $N \bowtie^d I$ and let $abm \in N$ for non-unit elements $a, b \in R$ and $m \in M$. Then $(a, a)(b, b)(m, m) \in N \bowtie^d I$. Since $(a, a), (b, b)$ are non-units and $N \bowtie^d I$ is an $S \bowtie I$-1-absorbing primary submodule of $M \bowtie^d I$, we have either

$$(s, s+i)(a, a)(b, b) \in (N \bowtie^d I :_{R\bowtie I} M \bowtie^d I)$$

or

$$(s, s+i)(m, m) \in M \bowtie^d I - \mathrm{rad}(N \bowtie^d I).$$

In the first case, we have $sab \in (N :_R M)$ by Lemma 2.6 (b). In the second case, $M \bowtie^d I\text{-}rad(N \bowtie^d I) = M\text{-}rad(N) \bowtie I$ by Theorem 2.10 and we conclude that $(sm, (s+i)m) \in M\text{-}rad(N) \bowtie^d I$, and so $sm \in M\text{-}rad(N)$. If such an N does not exists, it is well known that $M\text{-}rad(N) = M$. Then N is an S-1-absorbing primary submodule of M. □

Remark 2.12. For an ideal $0 \times I$ of the ring $R \bowtie I$ we have $(0 \times I)M \bowtie^d I = \langle \{(0, i)(m_1, m_2) | i \in I, (m_1, m_2) \in M \times M \text{ with } m_1 - m_2 \in (0 :_M Ann_R(I))\} \rangle$. Also, we have $(0 :_M Ann_R(I)) \times (0 :_M Ann_R(I))$ is an $R \bowtie I$-submodule of an R-module $M \bowtie^d I$.

Proposition 2.13. We have the following isomorphisms (of R and $R \bowtie I$ modules):

(a) $M \bowtie^d I/0 \times (0 :_M Ann_R(I)) \cong M$.
(b) $M \bowtie^d I/(0 :_M Ann_R(I)) \times (0 :_M Ann_R(I)) \cong M/(0 :_M Ann_R(I))$.

Proof. One can see that the morphisms of $R \bowtie I$-modules (resp., of R-modules) $\Phi : M \bowtie^d I \to M$; $(m_1, m_2) \to m_1$ and $\Psi : M \bowtie^d I \to M/(0 :_M Ann_R(I))$; $(m_1, m_2) \to \overline{m_1}$ are surjective with $ker(\Phi) = 0 \times (0 :_M Ann_R(I))$ and $ker(\Psi) = (0 :_M Ann_R(I)) \times (0 :_M Ann_R(I))$. Hence, we have the desired isomorphisms. □

Remark 2.14. If R-modules are regarded as $R \bowtie I$-modules via the second projection $R \bowtie I \to R$; $(r, r+i) \to r+i$, then with a similar proof as in the Proposition 2.13, we obtain $M \bowtie^d I/(0:_M Ann_R(I)) \times 0 \cong M$ (isomorphism of R and $R \bowtie I$ modules).

Proposition 2.15. The $R \bowtie I$-module $M \bowtie^d I$ is Noetherian (resp., Artinian) if and only if the R-module M is Noetherian (resp., Artinian).

Proof. If the $R \bowtie I$-module $M \bowtie^d I$ is Noetherian (resp., Artinian) then, by Proposition 2.13, M is Noetherian (resp., Artinian) as an $R \bowtie I$-module. Thus, using [3, Lemma 2.3], M is Noetherian (resp., Artinian) as an R-module.

Now, suppose that M is a Noetherian (resp., Artinian), and so Noetherian (resp., Artinian) as $R \bowtie I$-module (by [3, Lemma 2.3]). Hence, using Proposition 2.13, to prove that $M \bowtie^d I$ is a Noetherian (resp., Artinian) $R \bowtie I$-module, it suffices to prove the same fact for the $R \bowtie I$-module $0 \times (0:_M Ann_R(I))$. To do so, let

$$N_1 \subseteq N_2 \subseteq \cdots (resp., N_1 \supseteq N_2 \supseteq \cdots)$$

be an ascending chain (resp., a descending chain) of $R \bowtie I$-submodules of $0 \times (0:_M Ann_R(I))$. Set $N_i = 0 \times P_i$. Certainly, P_i are R-submodules of $(0:_M Ann_R(I))$ and form an ascending chain (resp., a descending chain). On the other hand, $(0:_M Ann_R(I))$ is Noetherian (resp., Artinian). Thus, there exists an integer n such that $P_n = P_{n+1} = P_{n+2} = \cdots$, and so $N_n = N_{n+1} = N_{n+2} = \cdots$, as needed. □

By letting $M = R$, in Proposition 2.15 we have the next corollary.

Corollary 2.16. The ring $R \bowtie^d I$ is Noetherian (resp., Artinian) if and only if R is Noetherian (resp., Artinian).

Lemma 2.17. Let I be an ideal of R and M be an R-module. Then we have the following isomorphism of R-modules:

$$Hom_{R\bowtie I}(R, M \bowtie^d I) = (0:_M Ann(I)) \oplus (0:_M I).$$

Proof. Consider $g \in Hom_{R\bowtie I}(R, M \bowtie^d I)$ and set $g(1) = (m_1, m_2)$. For each $i \in I$ we have

$$g(0) = g((0,i).1) = (0,i)g(1) = (0,i)(m_1,m_2) = (0, im_2).$$

Thus, $m_2 \in (0:_M I)$. Now, consider the following map:

$$\Psi : Hom_{R\bowtie I}(R, M \bowtie^d I) \to (0:_M Ann(I)) \oplus (0:_M I)$$

$$\Psi(g) = (m_1 - m_2, m_2) \text{ with } g(1) = (m_1, m_2).$$

It is easily seen that Ψ is an injective R-homomorphism of modules. Now, for each $(x,y) \in (0:_M Ann(I)) \oplus (0:_M I)$, let $g \in Hom_{R\bowtie I}(R, M \bowtie^d I)$ defined by $g(1) = (x+y, y)$. It is clear that $\Psi(g) = (x,y)$, and so Ψ is surjective. □

Remark 2.18. Note that in the above lemma, the $R \bowtie I$-module multiplication over R is the one fixed in the Introduction of this paper. However, using a similar proof, we can prove that the same isomorphism holds also when we consider the second $R \bowtie I$-module multiplication over R given by the morphism of rings $R \bowtie I \to R$, $(r, r+i) \mapsto r+i$.

Theorem 2.19. (a) *The $R \bowtie I$-module $M \bowtie^d I$ is injective if and only if $(0:_M Ann(I))$ and $(0:_M I)$ are injective R-modules.*

(b) *The $R \bowtie I$-module $M \bowtie^d I$ is projective (resp., flat) if and only if M is an R-module projective (resp., flat).*

Proof. The ring $R \bowtie I$ is obtained by the following pullback of rings [3, Theorem 3.3]:

$$R \bowtie I[d]^{p_2}[rr]^{p_1} R[d]^{\pi} R[rr]^{\pi} R/I$$

where $p1 : R \bowtie I \to R$; $(r, r+i) \mapsto r$ and $p2 : R \bowtie I \to R; (r, r+i) \mapsto r+i$, and π is the canonical projection of R onto R/I.

(a) This follows by using Lemma 2.17, Remark 2.18, and [8, Theorem 1].

(b) By using Proposition 2.13 and Remark 2.14, via p_1, we get that

$$M \bowtie^d I \otimes_{R\bowtie I} R \cong M \bowtie^d I \otimes_{R\bowtie I} \frac{R \bowtie I}{0 \times (0 :_M Ann_R(I))} \cong$$

$$\frac{M \bowtie^d I}{(0 \times (0 :_M Ann_R(I)))M \bowtie^d I} \cong \frac{M \bowtie^d I}{0 \times (0 :_M Ann_R(I))} \cong M.$$

and via p_2, we get that

$$M \bowtie^d I \otimes_{R\bowtie I} R \cong M \bowtie^d I \otimes_{R\bowtie I} \frac{R \bowtie I}{(0 :_M Ann_R(I)) \times 0} \cong$$

$$\frac{M \bowtie^d I}{((0 :_M Ann_R(I)) \times 0)M \bowtie^d I} \cong \frac{M \bowtie^d I}{(0 :_M Ann_R(I)) \times 0} \cong M.$$

Therefore, the result follows from [8, Theorem 1]. □

Example 2.20. [3, Example 3.4] For each positive integer k, the $\mathbb{Z} \bowtie k\mathbb{Z}$-module $\mathbb{Q} \bowtie^d k\mathbb{Z} = \mathbb{Q} \times \mathbb{Q}$ is flat and injective but not projective.

Proposition 2.21. Let I be a finitely generated ideal of R and M be a finitely generated R-module. Then $dim_{R\bowtie I}(M \bowtie^d I) = dim_R(M)$.

Proof. By [3, Lemma 3.6] and Lemma 2.6, we have that $Ann_{R\bowtie I}(M \bowtie I) = Ann_{R\bowtie I}(M \bowtie^d I)$. Now the result follows from [3, Lemma 3.7]. □

Notation 2.22. Let I be an ideal of R and M be an R-module. Set:
$T_1 = \{(0, i) : i \in I \setminus \{0\}\}$;
$T_2 = \{(a, a+i) : a \in Zd_R(M) \setminus \{0\}, i \in I\}$;
$T_3 = \{(a, a+i) : a \in R \setminus \{0\}, i \in I, (i+a)y = 0 for\ some\ 0 \neq y \in (0 :_M Ann_R(I))\}$.

Theorem 2.23. *Let I be an ideal of R and M be a non-zero R-module. Then $Zd_{R\bowtie I}(M \bowtie^d I) \setminus \{(0,0)\} = T_1 \cup T_2 \cup T_3$.*

Proof. First we show that $T_1 \cup T_2 \cup T_3 \subseteq Zd_{R\bowtie I}(M \bowtie^d I) \setminus \{(0,0)\}$. So, assume that $(0, i) \in T_1$. If we can consider $0 \neq x \in (0 :_M Ann_R(I))$. Then $(0, i)(x, 0) = (0, 0)$ implies that $(0, i) \in Zd_{R\bowtie I}(M \bowtie^d I) \setminus \{(0,0)\}$. If $(0 :_M Ann_R(I)) = 0$, then $MAnn_R(I) \neq 0$. So, there exists $x \in M$ and $b \in Ann_R(I)$ such that $0 \neq bx$. Then $(0, i)(bx, bx) = (0, 0)$ implies that $(0, i) \in Zd_{R\bowtie I}(M \bowtie^d I) \setminus \{(0,0)\}$. Thus, we have $T_1 \subseteq Zd_{R\bowtie I}(M \bowtie^d I) \setminus \{(0,0)\}$. Now, suppose that $(a, a+i) \in T_2$. Then there exists $0 \neq x \in M$ such that $ax = 0$. Consider the following two cases, namely the case where $x \in (0 :_M Ann_R(I)$ and the case $x \notin (0 :_M Ann_R(I)$.

Case 1. Assume that $x \in (0 :_M Ann_R(I)$. Then $(a, a+i)(x, 0) = (0, 0)$ implies that $(a, a+i) \in Zd_{R\bowtie I}(M \bowtie^d I) \setminus \{(0,0)\}$ and so $T_2 \subseteq Zd_{R\bowtie I}(M \bowtie^d I) \setminus \{(0,0)\}$.

Case 2. Suppose that $x \notin (0 :_M Ann_R(I)$. Then $xAnn_R(I) \neq 0$. Thus there exists $b \in Ann_R(I)$ such that $bx \neq 0$. Hence $(a, a+i)(bx, bx) = (0, 0)$. It follows that $(a, a+i) \in Zd_{R \bowtie I}(M \bowtie^d I) \setminus \{(0, 0)\}$. So, $T_2 \subseteq Zd_{R \bowtie I}(M \bowtie^d I) \setminus \{(0, 0)\}$. Now, let $(a, a+i) \in T_3$. Then $(a, a+i)(0, y) = (0, 0)$ implies that $T_3 \subseteq Zd_{R \bowtie I}(M \bowtie^d I) \setminus \{(0, 0)\}$. Therefore, $T_1 \cup T_2 \cup T_3 \subseteq Zd_{R \bowtie I}(M \bowtie^d I) \setminus \{(0, 0)\}$.

Now, let $(a, a+i) \in Zd_{R \bowtie I}(M \bowtie^d I) \setminus \{(0, 0)\}$. If $a = 0$, then $i \neq 0$ and $(a, a+i) = (0, i) \in T_1$. So, we are done. Thus assume that $a \neq 0$. There exists $(0, 0) \neq (x, y) \in M \bowtie^d I$ such that $(a, a+i)(x, y) = (0, 0)$. Consider the following two cases, namely the case where $x = 0$ and the case $x \neq 0$.

Case 1. Assume that $x \neq 0$. Then $ax = 0$ implies that $a \in Zd_R(M)$. Thus $(a, a+i) \in T_2$.

Case 2. Assume that $x = 0$. Then $(x, y) \neq (0, 0)$ implies that $y = y - 0 = y - x \in (0 :_M Ann_R(I)$. Now, $(a, a+i)y = 0$ implies that $(a, a+i) \in T_3$. Therefore, $Zd_{R \bowtie I}(M \bowtie^d I) \setminus \{(0, 0)\} \subseteq T_1 \cup T_2 \cup T_3$, as needed. □

Lemma 2.24. Let I be an ideal of R, M be an R-module such that $I \subseteq Zd_R(M)$, and let $Zd_R(M)$ be an ideal of R. Then $Zd_{R \bowtie I}(M \bowtie^d I) \setminus \{(0, 0)\} = T_1 \cup T_2$.

Proof. By Theorem 2.23, it is enough to show that $T_3 \subseteq T_2$. So, let $(a, a+i) \in T_3$. Then for some $0 \neq y \in (0 :_M Ann_R(I))$ we have $(i + a)y = 0$. It follows that $i + a \in Zd_R(M)$. Now since $i \in I \subseteq Zd_R(M)$ and $Zd_R(M)$ is an ideal of R, we have that $a \in Zd_R(M)$. Thus $(a, a+i) \in T_2$, as needed. □

Theorem 2.25. *Let I be an ideal of R and M be an R-module. Then we have the following.*

(a) *If $(Zd_{R \bowtie I}(M \bowtie^d I))^2 = 0$, then $(Zd_R(M))^2 = 0$ and $I^2 = 0$.*
(b) *If $I^2 = 0$, $(Zd_R(M))^2 = 0$, and $Zd_R(M)$ is an ideal of R, then $(Zd_{R \bowtie I}(M \bowtie^d I))^2 = 0$.*

Proof. (a) Let $(Zd_{R \bowtie I}(M \bowtie^d I))^2 = 0$ and $0 \neq a \in Zd_R(M)$, $0 \neq b \in Zd_R(M)$. Then $(a, a), (b, b) \in T_2 \subseteq Zd_{R \bowtie I}(M \bowtie^d I)$. Thus $(a, a)(b, b) = (0, 0)$. Hence $ab = 0$ and so $(Zd_R(M)^2 = 0$. If $0 \neq i \in I$ and $0 \neq j \in I$, then $(0, i), (0, j) \in T_1 \subseteq Zd_{R \bowtie I}(M \bowtie^d I)$. Thus $(0, i)(0, j) = (0, 0)$. It follows that $ij = 0$. Therefore, $I^2 = 0$.

(b) Let $I^2 = 0$, $(Zd_R(M))^2 = 0$, and $Zd_R(M)$ be an ideal of R. We claim that $I^2 = 0$ implies $I \subseteq Zd_R(M)$. To see this, let $a \in I$ and $0 \neq m \in M$. If $am = 0$, then $a \in Zd_R(M)$. If $am \neq 0$, then $I^2 = 0$ implies that $a(am) = 0$. Thus $a \in Zd_R(M)$. Therefore, $I \subseteq Zd_R(M)$. Now, by Lemma 2.24, $Zd_{R \bowtie I}(M \bowtie^d I) \setminus \{(0, 0)\} = T_1 \cup T_2$. Assume that $(a_1, b_1), (a_2, b_2) \in Zd_{R \bowtie I}(M \bowtie^d I)$. Then $(a_1, b_1), (a_2, b_2) \in T_1$ or $(a_1, b_1), (a_2, b_2) \in T_2$ or $(a_1, b_1) \in T_1$, $(a_2, b_2) \in T_2$. If $(a_1, b_1) \in T_1$ and $(a_2, b_2) \in T_2$. Then $a_1 = 0$, $0 \neq b_1 \in I$, $0 \neq a_2 \in Zd_R(M)$, and $b_2 = a_2 + i$ for some $i \in I$. Thus $(a_1, b_1)(a_2, b_2) = (0, b_1(a_2 + i))$. We have $b_1 \in I \subseteq Zd_R(M)$ and $a_2 \in Zd_R(M)$. So, $b_1 a_2 \in (Zd_R(M))^2 = 0$ and $b_1 i \in I^2 = 0$. Therefore, $(a_1, b_1)(a_2, b_2) = 0$. Also, in other case, it is easy to see that $(a_1, b_1)(a_2, b_2) = 0$, as needed. □

An R-module M is called a *reduced module* if $r^2 m = 0$ implies that $rm = 0$, where $r \in R$ and $m \in M$ [9].

Proposition 2.26. Let I be an ideal of R and M be an R-module. Then $M \bowtie^d I$ is reduced $R \bowtie I$-module if and only if M is a reduced R-module.

Proof. This is straightforward. □

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References

[1] H. Ansari-Toroghy and F. Farshadifar, *The dual notion of multiplication modules.* Taiwanese J. Math. **11** (4) (2007), 1189–1201.

[2] A. Barnard, *Multiplication modules.* J. Algebra, **71** (1) (1981), 174-178.

[3] E.M. Bouba, N. Mahdou and M. Tamekkante, *Duplication of a module along an ideal.* Acta Math. Hungar., **154** (1) (2018), 29–42.

[4] J. Dauns, *Prime modules*, J. Reine Angew. Math., **298** (1978), 156–181.

[5] F. Farshadifar, *Survey on second submodules of modules over commutative rings*, https://doi.org/10.48550/arXiv.2411.09333.

[6] E.H. Feller and E.W. Swokowski, *Prime modules.* Canadian J. Math., **17** (1965), 1041–1052.

[7] M. Issoual, N. Mahdou, N.A. Ozkirisci and E.Y. Celikel, *On S-1-absorbing primary submodules.* arXiv preprint arXiv:2203.04690, 2022.

[8] N.V. Kasmatov, *Bounds for the homological dimensions of pullbacks*, J. Math. Sci. (N.Y.), **112** (2002), 4367-4370.

[9] T. K. Lee and Y. Zhou, *Reduced modules*, In Rings, modules, algebras, and abelian groups, volume 236 of Lecture Notes in Pure and Appl. Math., pages 365-377. Dekker, New York, 2004.

[10] E. A. Ugurlu, *Generalizations of prime submodules over non-commutative rings*, Journal of Hyperstructures, **11** (1) (2023), 65-83.

[11] S. Yassemi, *Coassociated primes of modules over commutative rings*, Math. Scand., **10** (1997), 175–187.

[12] S. Yassemi, *The dual notion of prime submodules.* Arch. Math. (Brno), **37** (4) (2001), 273–278.