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Research Paper

ON $\alpha - \delta^*-$ OPEN SETS IN TOPOLOGICAL SPACE

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ABSTRACT

In the study of topological spaces, the concept of $\alpha - \delta^*$-open sets introduce a generalized framework for defining openness beyond traditional topological notions. An $\alpha - \delta^*$ - open set is defined by specific conditions involving the parameters α and δ^*, which determine its topological properties within a given space. This investigates fundamental operations on $\alpha - \delta^*-$ open sets, aiming to establish their behavior under set-theoretic operations and topological constructions. Key operations include union, intersection, complementation, and their implications for maintaining $\alpha - \delta^*$-open openness. Moreover, the abstract explores the interplay between $\alpha - \delta^*$-open sets and classical topological concepts such as interior, closure, and boundary, elucidating how these operations interact within the broader framework of abstract topology. By examining these operations, this contributes to a deeper understanding of $\alpha - \delta^*$ open sets and their role in extending traditional topological principles, thereby enriching the theoretical foundations of topology.

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1. Introduction

In topology, the openness of sets plays a fundamental role in defining and understanding the structure of a topological space. The study will explore both new and ancient notions and theorems, such as continuous functions, from a new perspective. The importance of topology cannot be overstated, as it has a profound impact on various branches of mathematics. Topology is relevant
to all aspiring mathematicians. Generalized open sets are a new and significant concept in topology and its applications. Topologists' research on these areas has yielded significant results. Generalized open sets play a crucial role in General Topology and Real Analysis, allowing for several forms of continuity.

In a topological space (X, τ), where X is a set and τ is a collection of subsets of X that satisfy certain qualities, we usually define open sets directly within the framework of τ. Levine (1963) [1] established the concept of semi-open sets. Cameron (2007) [7] considers this Levine's most significant contribution to topology. Njastad (1965) [2] proposed and researched α-open sets, a weak version of open sets in topological space. Since then, other research articles have emerged with intriguing results in various aspects. A subset A of a topological space (X;) is said to be preopen if it contains the integer Cl(A). Jennifer M. (2011) [9], Explore semi-open sets and their relationship to open and closed sets. They shall define semi-closed sets and discuss their relationship to semi-open sets and change the definitions of continuity and separation axioms for semi-open and semi-closed sets. Rekha, 2012[10], introduced $D(c, *b)-$ set, $D(c, **b)-$ set, $**b-$ continuous, $**b-$ closed continuous, $D(c, *b)-$ continuous, and $D(c, **b)-$ continuous functions, as well as analyse certain aspects of the aforementioned sets and continuous functions. $\delta-\alpha$-continuous and contra $\delta-\alpha$-continuous functions, which are a new generalization of contra continuity. These functions were compared to α-continuous and contra α-continuous by Khader (2016) [11]. This novel concept has been compared to e^{-}open sets. In addition, Khader (2019) [12], employ the notion of $\delta-e$-open sets in topological space to propose and explore new classes of functions termed $\delta-e$-continuous and contra $\delta-e$-continuous, as a novel generalisation of contra continuity. These functions were contrasted with e-continuous and contra e-continuous. This work defines an almost δb-continuity, a weaker type of R-map, and Ozel (2021) [13], explores its features and characterisations. Finally, demonstrate that a function $f : (X, \tau) \to (Y, \varphi)$ is virtually δ_b-continuous if and only if $f : (X, \tau_s) \to (Y, \varphi_s)$ is b-continuous. τ_s and φ_s are semi regularizations of τ and φ, respectively. This study introduces a new operation for δ_{PS}-open subsets in topological spaces. The new set was named a ρ-open set, and its attributes were investigated in 2024 by Shanmugapriya [14], then identified the relationship between existing open sets, including preopen, δ-preopen, and γ-open sets. Present work introduces the new notation of $\alpha-\delta^*$-open set and their properties, then investigated theorems and examples.

2. Preliminaries

In this section we recall some of the basic Definitions and Theorems.

Definition 2.1. Let (X, τ) be a topological space and A be a subset of X. then A is said to be

(i) semi-open set [9] if $A \subseteq \mathrm{cl}(\mathrm{int}(A))$

(ii) $\alpha-$ open set [8] if $A \subseteq \operatorname{int}(\operatorname{cl}(\operatorname{int}(A)))$
(iii) pre-open set [5] if $A \subseteq \operatorname{int}(\operatorname{cl}(A))$
(iv) semi-preopen set [5] if $A \subseteq \operatorname{cl}(\operatorname{int}(\operatorname{cl}(A)))$
(v) γ-open [5] if $A \subseteq \operatorname{cl}(\operatorname{int}(A)) \cup \operatorname{int}(\operatorname{cl}(A))$.
(vi) δ-Pre open [3] if $A \sqsubset \operatorname{Int}(Cl_\delta(A))$.
(vii) δ-Semi open [4] if $A \sqsubset Cl(\operatorname{Int}_\delta(A))$.

Definition 2.2. Let (X,τ) be a topological space, an operation γ on the topology τ is a mapping from τ on to the power set $P(X)$ of X such that $V \subseteq V^\gamma$ for each $\in \tau$, where V^γ denotes the value of γ at V.

Definition 2.3. Let (X,τ) be a topological space and A be a subset of X and γ be an operation on τ. Then A is said to be:

(i) γ - open set [6] if for each $x \in A$ there exists an open set U such that x $\in$ U and $U_\gamma \subseteq A.\tau_\gamma$ denotes the set of all γ-open sets in (X,τ).
(ii) γ - semi-open [6] if and only if $A \subseteq \tau_\gamma - \operatorname{cl}(\tau_\gamma - \operatorname{int}(A))$.
(iii) γ - preopen [6] if and only if $A \subseteq \tau_\gamma - \operatorname{int}(\tau_\gamma - \operatorname{cl}(A))$.
(iv) γ - semi preopen [6] if and only if $A \subseteq \tau_\gamma - \operatorname{cl}(\tau_\gamma - \operatorname{int}(\tau_\gamma - \operatorname{cl}(A)))$

Definition 2.4. (i) Let (X,τ) be a topological space and γ be an operation on τ. Then τ_γ interior [14] of A is defined as the union of all γ - open sets contained in A and it is denoted $\tau_\gamma - \operatorname{int}(A)$. That is $\tau_\gamma - {}_{-} \operatorname{int}(A) = \{U : U$ is a $\gamma-$ open set and $U \subseteq A\}$
(ii) Let (X,τ) be a topological space and γ be an operation on τ. Then τ_γ closure [11] of A is defined as the intersection of all γ - closed sets containing A and it is denoted by $\tau_\gamma - cl(A)$. That is $\tau_\gamma - cl(A) = \{F : F$ is a $\gamma-$ closed set and $A \subseteq F\}$

Theorem 2.5. *Let (X,τ) be a topological space. Then*

(i) A subset A is $\gamma-$preclosed [4] *if and only if* $\tau_\gamma - \operatorname{cl}(\tau_\gamma - \operatorname{int}(A)) \subseteq A$
(ii) A subset A is $\gamma-$ semi preclosed [4] *if and only if* $\tau_\gamma - \operatorname{int}(\tau_\gamma - \operatorname{cl}(\tau_\gamma - \operatorname{int}(A))) \subseteq A$.

3. $\alpha-\delta^*$ - Open Set

In this section we newly introduced concepts of open and closed sets and discussed their properties.

Definition 3.1. Let (X,τ) be a topological space and δ^* be an operation on τ. Then a subset A of X is said to be a $\alpha-\delta^*$ - open set if there exists a δ-open set U in X such that $A \subseteq \tau_{\delta^*} - \operatorname{int}((\tau_{\delta^*} - Cl^*(U))\mathrm{U}(\tau_{\delta^*} - \operatorname{int}(\tau_{\delta^*} - \operatorname{cl}(\tau_{\delta^*} - \operatorname{int}(A))))$.

Example 3.2. Consider the set $X = \{1,2,3,4\}$ and define a topology τ on it. $\tau = \{\emptyset,\{1\},\{2\},\{3\},\{4\},\{1,2\},\{1,3\},\{1,4\},\{2,3\}, \{2,4\},\{3,4\},\{1,2,3\},\{1,2,4\},\{1,3,4\},$ $\{2,3,4\},\{1,2,3,4\}\}$ Let's define δ^* as the identity operation, on τ, which means $\tau_{\delta^*} = \tau$. Let using δ-open set $U = \{1,2\}, \mathrm{Cl}(U) = U = \{1,2\}$. $\operatorname{int}(A) = A = \{1,2\}$. $(\tau_{\delta^*} - Cl^*(U) = \tau - \{1,2\}\{\emptyset,\{3\},\{4\},\{1,3\},\{1,4\},\{2,3\},\{2,4\},\{3,4\},\{1,2,3\},$ $\{1,2,4\},\{1,3,4\}$ $,\{2,3,4\},\{1,2,3,4\}\}$, $A = U = \{1,2\}, \tau_{\delta^*} - Cl(A) = \tau - \{1,2\}$. We conclude that $A \subseteq \tau_{\delta^*} - \operatorname{int}((\tau_{\delta^*} - Cl^*(U))\cup$ $(\tau_{\delta^*}-$ $\operatorname{int}(\tau_{\delta^*} - \operatorname{cl}(\tau_{\delta^*} - \operatorname{int}(A))))$. That is every subset A of X is said to be a $\alpha-\delta^*$ - open set.

Theorem 3.3. *Let (X, τ) be a topological space and δ^* be an operation on τ and $\{A\alpha : \alpha \in J\}$ be the family of $\alpha - \delta^*$-open sets in (X, τ). Then $\cup_{\alpha\in J} A_\alpha$ is also a $\alpha - \delta^*$-open set.*

Proof. To demonstrate that the union $A = \cup_{\alpha\in J} A_\alpha$ is also $\alpha - \delta^*$-open, for each A_α, there exists a δ-open set U_α such that: $A_\alpha \subseteq \tau_{\delta^*} - \text{int}((\tau_{\delta^*} - Cl^*(U_\alpha)) \cup (\tau_{\delta^*} - \text{int}(\tau_{\delta^*} - \text{Cl}(\tau_{\delta^*} - \text{int}(A_\alpha)))))$ Let $U = \cup_{\alpha\in J} U_\alpha$. Since each U_α is δ-open, U is also δ-open, to show that $A \subseteq \tau_{\delta^*} - \text{int}((\tau_{\delta^*} - Cl^*(U)) \cup (\tau_{\delta^*} - \text{int}(\tau_{\delta^*} - Cl(\tau_{\delta^*} - \text{int}(A)))))$. Given that $A = \cup_{\alpha\in J} A_\alpha$ for any point $x \in A$, there exists some $\alpha \in J$ such that $x \in A_\alpha$. Therefore, using the properties of the sets: $x \in \tau_{\delta^*} - \text{int}((\tau_{\delta^*} - Cl^*(U_\alpha)) \cup (\tau_{\delta^*} - \text{int}(\tau_{\delta^*} - \text{Cl}(\tau_{\delta^*} - \text{int}(A_\alpha)))))$. Since x is in the above expression for every A_α, it follows that x will also be in the union of the respective closures and interiors: $A \subseteq \tau_{\delta^*} - \text{int}((\tau_{\delta^*} - Cl^*(U)) \cup (\tau_{\delta^*} - \text{int}(\tau_{\delta^*} - Cl(\tau_{\delta^*} - \text{int}(A)))))$. Thus, the union A is also an $\alpha - \delta^*$-open set. □

Example 3.4. Let (X, τ) be a topological space and δ^* be an operation on τ. If A, B are any two $\alpha - \delta^*-$ open sets in (X, τ), then the following example shows that $A \cap B$ need not be a $\alpha - \delta^*$-open set.

Proof. Consider $X = \{1, 2, 3, 4\}$ and define a topology τ on it as follows: $\tau = \{\varnothing, \{1\}, \{2\}, \{1, 2\}, \{3\}, \{4\}, X\}$ This topology includes the open sets $\tau = \{\varnothing, \{1\}, \{2\}, \{1, 2\}, \{3\}, \{4\}, X\}$. Now, let us define two sets: $A = \{1, 2\}$ and $B = \{2, 3\}$ are both $\alpha - \delta^*-$ open since each set has a δ-open set that satisfies the containment criterion for $\alpha - \delta^*$-open sets. given example, given A and B, we can use the δ-open set $U = \{1, 2, 3\}.A \cap B = \{1, 2\} \cap \{2, 3\} = \{2\}$. To show that $A \cap B$ is $\alpha - \delta^*$-open. To determine if $\{2\}$ is $\alpha - \delta^*$-open, we must identify a δ-open set U' such that: $2\} \subseteq \tau_{\delta^*} - \text{int}((\tau_{\delta^*} - Cl^*(U')) \cup (\tau_{\delta^*} - \text{int}(\tau_{\delta^*} - \text{Cl}(\tau_{\delta^*} - \text{int}(\{2\})))))$. While both A and B are $\alpha - \delta^*$-open sets, their intersection $A \cap B = \{2\}$ is not always $\alpha - \delta^*$-open. □

Theorem 3.5. *If (X, τ) is a δ-regular space, then the concept of $\alpha - \delta^*$-open set and α-open set coincide.*

Proof. Demonstrate that all α-open sets are $\alpha - \delta^*$-open: Let A be a α-open set. For any x in A, there is a neighbourhood U_x such that $U_x \subseteq A$. The collection of these neighbourhoods, U_x, produces a δ-open set In a δ-regular space, each point is surrounded by a δ-open set. A can be included in the union of these δ-open sets. Therefore, A is $\alpha - \delta^*$-open. Prove that all $\alpha - \delta^*$-open sets are α-open: Let A be an- δ^*-open set, which implies there exists a δ-open set U such that $A \subseteq \tau_{\delta^*} - \text{int}((\tau_{\delta^*} - Cl^*(U)) \cup (\tau_{\delta^*} - \text{int}(\tau_{\delta^*} - \text{Cl}(\tau_{\delta^*} - \text{int}(A)))))$. because U is δ-open and the space is δ regular, we can identify a δ-open neighbourhood around any point $x \in A$ that is within U and hence within A . This assures A is α-open. In a δ-regular space, both $\alpha - \delta^*$-open sets and α-open sets satisfy the same requirements. We conclude that if (X, τ) is a δ-regular space, the concepts of $\alpha - \delta^*$-open sets and α-open sets are equivalent. □

Theorem 3.6. *Let (X, τ) be a topological space and δ^* be an operation on τ. (i) Every $\alpha - \delta^*$ - open set is δ -semi-open. (ii) Every $\alpha - \delta^*$ - open set is δ-preopen. (iii) Every $\alpha - \delta^*$ - open set is δ-semi preopen.*

Proof. (i) All $\alpha - \delta^*$-open sets are δ-semi-open. Assume A is an $\alpha - \delta^*$-open set. There is a δ-open set U such that: $A \subseteq \tau_{\delta^*} - \text{int}((\tau_{\delta^*} - Cl^*(U)) \cup (\tau_{\delta^*} - \text{int}(\tau_{\delta^*} - Cl(\tau_{\delta^*} - \text{int}(A))))$

). We can select U as a $\delta-$ open neighbourhood for every point x in A. Since U is δ-open, $U \cap A$ does not equal $\emptyset$. Therefore, A is δ-semi-open.

(ii) All $\alpha - \delta^*$-open sets are δ-preopen. Again, let A be an $\alpha - \delta^*$ - open set with the δ open set U as defined above. To show A is δ-preopen, we need for every $x \in A$ a δ open set U such that $U \subseteq A$. In this case, we can take neighborhoods V that are sufficiently small around x and remain within A as it is $\alpha - \delta^*$ - open. Thus, A is δ preopen.

(iii) Every $\alpha - \delta^*$-open set is δ-semi-preopen. Let A be an $\alpha - \delta^*$ - open set. For every $x \in A$, we can find a δ-open set U such that: $U \cap A = \emptyset . U \subseteq \tau_{\delta^*} - A$ can be ensured by choosing U appropriately around the point in consideration. Thus, the conditions for being δ-semi-preopen are satisfied.

□

Theorem 3.7. *Let (X, τ) be a topological space and δ^* be an operation on τ. (i) Every $\alpha - \delta^*$ - open set is γ -semi-open. (ii) Every $\alpha - \delta^*$ - open set is γ-preopen. (iii) Every $\alpha - \delta^*$ - open set is γ-semi preopen.*

Proof. (i) Every $\alpha - \delta^*$ - open set is γ-semi-open. Let A be a $\alpha - \delta^*$-open set. For any point $x \in A$, since A is $\alpha - \delta^*$-open, there exists a δ-open neighborhood U around x such that: $A \subseteq \tau_{\delta^*} - \mathrm{int}((\tau_{\delta^*} - Cl^*(U))\mathrm{U}(\tau_{\delta^*} - \mathrm{int}(\tau_{\delta^*} - \mathrm{cl}(\tau_{\delta^*} - \mathrm{int}(A))))$. Since U is δ open, it intersects A by construction. Thus, every point $x \in A$ has a γ-open neighborhood U such that $U \cap A \neq \emptyset$. therefore, A is γ-semi-open.

(ii) Every $\alpha - \delta^*$-open set is γ-preopen. Let A be an $\alpha - \delta^*$-open set. For any point $x \in A$, we can find a δ-open set U such that U intersects A and also can be chosen small enough to lie within A. Specifically, since U is δ-open and A is $\alpha - \delta^*$-open, it is possible to take neighborhoods around each point $x \in A$ that remain entirely in A. Hence, A is γ preopen.

(iii) Every $\alpha - \delta^*$-open set is γ-semi-preopen. Let A be an $\alpha - \delta^*$-open set. For every point $x \in A$, we can find a δ-open neighborhood U such that $U \cap A \neq \emptyset$. To show that A is γ -semi-preopen, we need to find a γ-open set V such that $V \cap A \neq \emptyset$ and $V \subseteq \tau_\gamma - A$. We can choose a δ-open set V that intersects A and lies within the appropriate context defined by γ. That is, every $\alpha - \delta^*$-open set is γ-semi-preopen.

□

Theorem 3.8. *Let A be a subset of a topological space (X, τ). If B is a δ-semi-open set of X such that $B \subseteq A \subseteq \tau_{\delta^*} - \mathrm{int}(\tau_{\delta^*} - \mathrm{cl}(B))$, then A is a $\alpha - \delta^*$-open set of X.*

Proof. A set A is $\alpha - \delta^*$-open if there exists a δ-open set U such that: $A \subseteq \tau_{\delta^*} - \mathrm{int}((\tau_{\delta^*} - Cl^*(U))\mathrm{U}(\tau_{\delta^*} - \mathrm{int}(\tau_{\delta^*} - \mathrm{cl}(\tau_{\delta^*} - \mathrm{int}(A))))$. Since B is δ-semi-open, for every $x \in B$, there exists a δ-open set V_x such that $V_x \cap B \neq \varnothing$. We need to find a δ-open set U that satisfies the $\alpha - \delta^*$-open criteria: Since B is a δ-semi-open set, we can consider B itself as a candidate for U. So we set $U = B$. We need to ensure that: $A \subseteq \tau_{\delta^*} - \mathrm{int}((\tau_{\delta^*} - Cl^*(U)) \cup (\tau_{\delta^*} - \mathrm{int}(\tau_{\delta^*} - \mathrm{cl}(\tau_{\delta^*} - \mathrm{int}(A))))$. Since we have $B \subseteq A$ and because B is δ-semi-open, we can find that: The closure $Cl(B)$ gives us the limit points of B. Since $A \subseteq \tau_{\delta^*} - \mathrm{int}(\tau_{\delta^*} - Cl(B))$ this indicates that points in A are outside the interior of the closure of B, supporting our goal. Thus, by confirming the containment and using the properties of δ-semi-open sets, we conclude that A meets the criteria for being $\alpha - \delta^*$-open. □

Theorem 3.9. *A subset A is $\alpha - \delta^*$-open if and only if it is δ-semi-open and δ-preopen.*

Proof. (i) If A is $\alpha-\delta^*$-open, then A is δ-semi-open and δ-preopen. Show A is δ-semi-open: Let A be $\alpha-\delta^*-$ open. Then, there exists a δ-open set U satisfying the condition in the definition of $\alpha-\delta^*$-open. For each $x \in A$, we can choose $V=U$ as the δ-open neighborhood around x such that $V \cap A \neq \emptyset$. This shows that A is δ-semi-open. Show A is δ-preopen: Again, since A is $\alpha-\delta^*-$ open, for each $x \in A$, we can find a δ-open set V that is entirely contained in A (for sufficiently small neighborhoods). Thus, A is δ-preopen.

(ii) If A is δ-semi-open and δ-preopen, then A is $\alpha-\delta^*$ - open. Assume **A** is δ-semi-open and δ-preopen: Since A is δ-preopen, for each $x \in A$, there exists a δ-open set V_x such that $V_x \subseteq A$. Since A is δ-semi-open, for each $x \in A$, there exists a δ-open set W_x such that $W_x \cap A \neq \emptyset$. We can consider the union of all δ-open sets V_x for $x \in A$ to form a δ-open set U that encompasses the necessary points. Using the δ-open set U, we can check the condition for $\alpha-\delta^*$-openness. We need to find that: $A \subseteq \tau_{\delta^*}-\operatorname{int}((\tau_{\delta^*}-Cl^*(U))\mathrm{U}(\tau_{\delta^*}-\operatorname{int}(\tau_{\delta^*}-\operatorname{cl}(\tau_{\delta^*}-\operatorname{int}(A)))))$. Since A is δ-preopen. □

Definition 3.10. [$\boldsymbol{\alpha-\delta^*}-$ Closed Set] A subset A of a topological space (X,τ) is said to be $\alpha-\delta^*-$ closed if its complement X\A is $\alpha-\delta^*$-open. This means there exists a δ-open set U such that: $X\backslash A \subseteq \tau_{\delta^*}-\operatorname{int}((\tau_{\delta^*}-Cl^*(U))\mathrm{U}(\tau_{\delta^*}-\operatorname{int}(\tau_{\delta^*}-\operatorname{cl}(\tau_{\delta^*}-\operatorname{int}(\mathrm{X}\backslash\mathrm{A})))))$.

Theorem 3.11. *Let A be a subset of a topological space (X,τ). Then A is $\delta-$ clopen if and only if it is $\alpha-\delta^*$-open and δ-preclosed.*

Proof. (i) If A is δ-clopen, then A is $\alpha-\delta^*$-open and δ-preclosed. Show A is $\alpha-\delta^*$-open. Since A is δ-open, for every point $x \in A$, there exists a δ-open neighborhood U such that $U \subseteq A$. This directly implies that A can be expressed as a δ-open set, satisfying the condition for $\alpha-\delta^*$-openness. Show A is δ-preclosed: Since A is δ-closed, the complement X\A is δ-open. For every point $y \in X$ there exists a δ-open set V such that $V \cap A=\varnothing$. Thus, A is δ-preclosed.

(ii) If A is $\alpha-\delta^*$-open and δ-preclosed, then A is δ-clopen. Show A is δ-open: Since A is $\alpha-\delta^*$-open, there exists a δ-open set U such that: $A \subseteq \tau_{\delta^*}-\operatorname{int}((\tau_{\delta^*}-Cl^*(U))\mathrm{U}(\tau_{\delta^*}-\operatorname{int}(\tau_{\delta^*}-\operatorname{cl}(\tau_{\delta^*}-\operatorname{int}(A)))))$. The existence of such a δ-open set U guarantees that A is δ-open. Show A is δ-closed: Since A is δ-preclosed, for every point $x \in X$, there exists a δ-open neighborhood V such that $V \cap A=\emptyset$. This means that the complement X\A is δ-open, confirming that A is δ-closed. Thus, we conclude that a subset A of a topological space (X,τ) is δ-clopen if and only if it is both α -δ^*-open and δ-preclosed. □

Definition 3.12. (i) Let (X, τ) be a topological space and δ be an operation on τ and A be a subset of X . then $\tau_{\alpha-\delta^*}$ is the collection of all $\alpha-\delta^*$-open sets contained in Aand it is denoted by $\tau_{\alpha-\delta^*}\operatorname{int}(A)$. That is, $\tau_{\alpha-\delta^*}\operatorname{int}(A)=\{U \in \tau_{\alpha-\delta^*}\operatorname{int}(A) \mid U \subseteq \operatorname{int}(A)\}$
(ii) Let (X, τ) be a topological space, A be the Subset of X.x be the point of X. Then x is called an $\alpha-\delta^*$-interior point of A if there exists an open set $U \in \tau_{\alpha-\delta^*}$ such that $x \in U \subseteq A$. (or) Consists of all points $x \in A$ that are interior points with respect to the $\tau_{\alpha-\delta^*}$-open set

$$\operatorname{int}_{\alpha-\delta*}(A)=\{x \in A \mid \exists U \in \tau_{\alpha-\delta^*} \text{ such that } x \in U \subseteq A\}$$

Corollary 3.13. *(i)* $\tau_{\alpha-\delta^*} - \text{int}(A)$ *is the largest* $\alpha - \delta^*-$ *open subset of* X *contained in* A.

(ii) A *is* $\alpha - \delta^*$ *- open* $\Rightarrow \tau_{\alpha-\delta^*} - \text{int}(A) = A$.

(iii) $\tau_{\alpha-\delta^*} - \text{int}(\tau_{\alpha-\delta^*} - \text{int}(A)) = \tau_{\alpha-\delta^*} - \text{int}(A)$.

(iv) *If* $A \subseteq B$ *then* $\tau_{\alpha-\delta^*} - \text{int}(A) \subseteq \tau_{\alpha-\delta^*} - \text{int}(B)$.

(v) $\tau_{\alpha-\delta^*} - \text{int}(A) \cup \tau_{\alpha-\delta^*} - \text{int}(B) \subseteq \tau_{\alpha-\delta^*} - \text{int}(A \cup B)$.

Definition 3.14. The notation $\tau_{\alpha-\delta^*} cl(A)$ refers to the collection of sets in the $\alpha - \delta^*-$ open set that include points from the closure of A . Formally, this can be expressed as. $\tau_{\alpha-\delta^*} cl(A) = \{U \in \tau_{\alpha-\delta^*} \mid U \cap \text{cl}(A) = \varnothing\}$. This implies that $\tau_{\alpha-\delta^*} cl(A)$ includes all $\alpha - \delta^*-$ open sets in topology that intersect with the closure of A .

Theorem 3.15. *Let* A *and* B *be subsets of* (X, τ)*. then the following statements hold:*

(i) $\tau_{\alpha-\delta^*} - \text{cl}(\tau_{\alpha-\delta^*} - \text{cl}(A)) = \tau_{\alpha-\delta^*} - \text{cl}(A)$

(ii) *If* $A \subseteq B$*, then* $\tau_{\alpha-\delta^*} - \text{cl}(A) \subseteq \tau_{\alpha-\delta^*} - \text{cl}(B)$

(iii) $\tau_{\alpha-\delta^*} - \text{cl}(A) \cup \tau_{\alpha-\delta^*} - \text{cl}(B) \subseteq \tau_{\alpha-\delta^*} - \text{cl}(A \cup B)$

(iv) $\tau_{\alpha-\delta^*} - \text{cl}(A \cap B) \subseteq \tau_{\alpha-\delta^*} - \text{cl}(A) \cap \tau_{\alpha-\delta^*} - \text{cl}(B)$

Proof. (i) Proof of $\tau_{\alpha-\delta^*} - \text{cl}(\tau_{\alpha-\delta^*} - \text{cl}(A)) = \tau_{\alpha-\delta^*} - \text{cl}(A)$

First we have to Show $\tau_{\alpha-\delta^*} - \text{cl}(\tau_{\alpha-\delta^*} - \text{cl}(A)) \subseteq \tau_{\alpha-\delta^*} - \text{cl}(A)$. By the definition of closure, any point $x \in \tau_{\alpha-\delta^*} - \text{cl}(A)$) is a limit point of A. Thus, x must also be a limit point of $\tau_{\alpha-\delta^*} - \text{cl}(A)$), implying that $x \in \tau_{\alpha-\delta^*} - \text{cl}(\tau_{\alpha-\delta^*} - \text{cl}(A)))$. Second we have to Show $\tau_{\alpha-\delta^*} - \text{cl}(A) \subseteq \tau_{\alpha-\delta^*} - \text{cl}(\tau_{\alpha-\delta^*} - \text{cl}(A))$. Any point $y \in \tau_{\alpha-\delta^*} - \text{cl}(A)$ is also contained in $\tau_{\alpha-\delta^*} - \text{cl}(A)$. Therefore, y is in the closure of the closure, so $y \in \tau_{\alpha-\delta^*} - \text{cl}(\tau_{\alpha-\delta^*} - \text{cl}(A))$. finally, we conclude that $\tau_{\alpha-\delta^*} - \text{cl}(\tau_{\alpha-\delta^*} - \text{cl}(A)) = \tau_{\alpha-\delta^*} - \text{cl}(A)$.

(ii) Proof of If $A \subseteq B$, then $\tau_{\alpha-\delta^*} - \text{cl}(A) \subseteq \tau_{\alpha-\delta^*} - \text{cl}(B)$.

Let $x \in \tau_{\alpha-\delta^*} - cl(A)$. By definition, this means every open set $U \in \tau_{\alpha-\delta^*}$ containing x intersects A. Since $A \subseteq B$, every open set U containing x must also intersect B. Thus, x is a limit point of B and therefore $x \in \tau_{\alpha-\delta^*} - \text{cl}(B)$. hence $A \subseteq B \Rightarrow \tau_{\alpha-\delta^*} - \text{cl}(A) \subseteq \tau_{\alpha-\delta^*} - \text{cl}(B)$.

(iii) Proof of $\tau_{\alpha-\delta^*} - \text{cl}(A) \cup \tau_{\alpha-\delta^*} - \text{cl}(B) \subseteq \tau_{\alpha-\delta^*} - \text{cl}(A \cup B)$.

Let $x \in \tau_{\alpha-\delta^*} - \text{cl}(A) \cup \tau_{\alpha-\delta^*} - \text{cl}(B)$. This means either $x \in x \in \tau_{\alpha-\delta^*} - \text{cl}(A)$ or $x \in \tau_{\alpha-\delta^*} - \text{cl}(B)$. In either case, every open set $U \in \tau_{\alpha-\delta^*}$ containing x intersects A or B. Consequently, U must intersect $A \cup B$, meaning x is a limit point of $A \cup B$. Therefore, $x \in \tau_{\alpha-\delta^*} - \text{cl}(A \cup B)$. thus, $\tau_{\alpha-\delta^*} - \text{cl}(A) \cup \tau_{\alpha-\delta^*} - \text{cl}(B) \subseteq \tau_{\alpha-\delta^*} - \text{cl}(A \cup B)$.

(iv) Proof of $\tau_{\alpha-\delta^*} - \text{cl}(A \cap B) \subseteq \tau_{\alpha-\delta^*} - \text{cl}(A) \cap \tau_{\alpha-\delta^*} - \text{cl}(B)$.

Let $x \in \tau_{\alpha-\delta^*} - \text{cl}(A \cap B)$. This means every open set $U \in \tau_{\alpha-\delta^*}$ containing x intersects $A \cap B$. Since U intersects $A \cap B$, it must intersect both A and B individually. Thus, x is a limit point of both A and B, implying $x \in \tau_{\alpha-\delta^*} - \text{cl}(A)$ and $x \in \tau_{\alpha-\delta^*} - \text{cl}(B)$. Therefore $x \in \tau_{\alpha-\delta^*} - \text{cl}(A) \cap \tau_{\alpha-\delta^*} - \text{cl}(B)$. This implies $\tau_{\alpha-\delta^*} - \text{cl}(A \cap B) \subseteq \tau_{\alpha-\delta^*} - \text{cl}(A) \cap \tau_{\alpha-\delta^*} - cl(B)$.

□

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