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Research Paper

CR-SLANT WARPED PRODUCT SUBMANIFOLDS IN NEARLY KENMOTSU MANIFOLD

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ABSTRACT

This paper investigates CR-slant warped product submanifolds of the form $B \times_f N_\theta$ within a nearly Kenmotsu manifold. Here, B is a CR-product submanifold, N_θ is a slant submanifold and f denotes the warping function. We derive an inequality that relates the squared norm of the second fundamental form to the warping function, considering the behavior of the structure vector field. Additionally, the cases of equality are also explored. Finally, we establish several geometrical consequences of our main theorem.

1. Introduction

The concept of warped products is well-established in both differential geometry and physics. In 1969, Bishop and O'Neill [7] introduced warped products as a tool for studying manifolds with negative curvature, extending the idea of Riemannian product manifolds. They defined these manifolds as follows: Let (B, g_1) and (F, g_2) be two Riemannian manifolds, and let f be a differentiable function on B. For the product manifold $B \times F$, with

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projections $\gamma_1 : B \times F \to B$ and $\gamma_2 : B \times F \to F$, the warped product of B and F, denoted by $N = B \times_f F$, is endowed with a Riemannian structure defined as follows [7]

$$g(X_1, Y_1) = g_1(\gamma_{1*}X_1, \gamma_{1*}Y_1) + (f \circ \gamma_1)^2 g_2(\gamma_{2*}X_1, \gamma_{2*}Y_1),$$

for any vector field $X_1, Y_1 \in \Gamma(TN)$ and $*$ denotes the tangent maps. A warped product manifold is considered trivial or simply a Riemannian product manifold, if the warping function f is constant. It is well known that, for a vector field X_1 on B and X_2 on F, the following holds [7]:

$$\nabla_{X_2}X_1 = \nabla_{X_1}X_2 = X_1(lnf)X_2, \tag{1.1}$$

where ∇ is the Levi-Civita connection on N. Additionally, it is well established that B is totally geodesic and F is totally umbilical in the warped product $B \times_f F$ [7, 12].

Chen introduced the concept of CR-warped products in Kaehler manifolds in [12], establishing results on their existence and deriving sharp inequalities for the second fundamental form in terms of the warping function. Since then, several studies have extended these results, exploring similar inequalities in almost Hermitian and almost contact metric manifolds [5, 9, 16].

Şahin [18] introduced the concept of CR-slant warped products, also known as skew CR-warped products, in Kähler manifolds, establishing the foundational framework for their study. Building on this, Chen et al. [13] extended the study to pointwise CR-slant warped product submanifolds in Kähler manifolds and established corresponding geometric inequalities. Subsequently, Uddin and Ullah [20] obtained related inequalities in the setting of nearly cosymplectic manifolds, demonstrating how the ambient almost contact structure affects the behavior of the second fundamental form. More recently, Alqahtani and Almudawi [4] investigated CR-slant warped product submanifolds of the form $B \times N_\theta$ in nearly trans-Sasakian manifolds, thereby expanding the scope of CR-slant geometry to broader classes of almost contact metric manifolds. Several other geometers have also contributed to the study of CR-slant warped product submanifolds, as seen in [3, 21, 23]. These developments collectively indicate a growing interest in examining how the curvature conditions and the ambient geometric structure shape the behavior of CR-slant warped product submanifolds.

On the other hand, Kenmotsu [14] introduced a distinguished class of almost contact metric manifolds, known as Kenmotsu manifolds, which have been fundamental in the study of contact geometry. Shukla [19] subsequently generalized this framework by defining nearly Kenmotsu manifolds, providing a broader setting to investigate geometric properties of submanifolds under curvature constraints. Since then, these manifolds have been widely studied [1, 2, 22], highlighting their rich geometric structure and motivating the present study of CR-slant warped product submanifolds within them.

Building on previous research, we investigate CR-slant warped product submanifolds of the form $B \times_f N_\theta$ within a nearly Kenmotsu manifold. Here, $B = N_T \times N_\perp$ represents the CR-product of invariant and anti-invariant submanifolds of $\tilde{M}$, N_θ is a slant submanifold, and f denotes the warping function. We establish a sharp estimate for the squared norm of the second fundamental form in terms of the warping function, considering two cases based on whether the structure vector field is tangent to the invariant or anti-invariant submanifold. Moreover, we examine the conditions under which equality in the derived inequality holds.

As part of geometric applications, we discuss several exceptional cases that generalize various inequalities.

2. Preliminaries

Let $\tilde{M}$ be a $2m+1$-dimensional almost contact manifold equipped with an almost contact structure (ϕ, ξ, η, g), where ϕ is a (1, 1)-tensor field, ξ is a characteristic vector field, η is a 1-form and g is a Riemannian metric satisfy the following conditions [8]:

$$\phi^2 X_1 = -X_1 + \eta(X_1)\xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta \circ \phi = 0, \tag{2.1}$$

$$g(\phi X_1, \phi Y_1) = g(X_1, Y_1) - \eta(X_1)\eta(Y_1), \tag{2.2}$$

$$g(\phi X_1, Y_1) = -g(X_1, \phi Y_1), \quad \eta(X_1) = g(X_1, \xi) \tag{2.3}$$

for all vector fields X_1 and Y_1 on $\tilde{M}$. Then the structure (ϕ, ξ, η, g) is said to be almost contact metric manifold.
An almost metric contact manifold $\tilde{M}$ is called a Kenmotsu manifold if [14]

$$(\tilde{\nabla}_{X_1}\phi)Y_1 = g(\phi X_1, Y_1)\xi - \eta(Y_1)\phi X_1, \tag{2.4}$$

$$\tilde{\nabla}_{X_1}\xi = X_1 - \eta(X_1)\xi, \tag{2.5}$$

for all vector fields X_1, Y_1 tangent to $\tilde{M}$, where $\tilde{\nabla}$ denotes the Levi-civita connection associated with the Riemannian metric g. Furthermore, an almost contact metric manifold $\tilde{M}$ with structure (ϕ, ξ, η, g) is said to be a nearly Kenmotsu manifold if [19]

$$(\tilde{\nabla}_{X_1}\phi)Y_1 + (\tilde{\nabla}_{Y_1}\phi)X_1 = -\eta(X_1)\phi Y_1 - \eta(Y_1)\phi X_1, \tag{2.6}$$

for any X_1, Y_1 tangent to $\tilde{M}$. The covariant derivative of the tensor field ϕ is given by

$$(\tilde{\nabla}_{X_1}\phi)Y_1 = \tilde{\nabla}_{X_1}\phi Y_1 - \phi\tilde{\nabla}_{X_1}Y_1. \tag{2.7}$$

Let N be an n-dimensional submanifold immersed in $\tilde{M}$ with induced metric g. The tangent and normal subspaces of N in $\tilde{M}$ are denoted by $\Gamma(TN)$ and $\Gamma(T^{\perp}N)$, respectively.
The Gauss and Weingarten formulas are given by [6]:

$$\tilde{\nabla}_{X_1}Y_1 = \nabla_{X_1}Y_1 + h(X_1, Y_1) \tag{2.8}$$

and

$$\tilde{\nabla}_{X_1}X_2 = -A_{X_2}X_1 + \nabla^{\perp}_{X_1}X_2 \tag{2.9}$$

for all X_1, $Y_1 \in \Gamma(TN)$ and $X_2 \in \Gamma(T^{\perp}N)$, where ∇ and $\nabla^{\perp}$ denote the induced connections on the tangent bundle TN and $T^{\perp}N$ of N, respectively.
The second fundamental form h and shape operator A are related by the following equation [6]:

$$g(A_{X_2}X_1, Y_1) = g(h(X_1, Y_1), X_2), \tag{2.10}$$

for any X_1, $Y_1 \in \Gamma(TN)$ and $X_2 \in \Gamma(T^{\perp}N)$. The mean curvature vector H of N is given by

$$H = \frac{1}{n}tr(h) = \frac{1}{n}\sum_{i=1}^{n} h(e_i, e_i), \tag{2.11}$$

where n is the dimension of N and $(e_1, e_2, ...e_n)$ is an local orthonormal frame of N.
A submanifold N is said to be totally umbilical if

$$h(X_1, Y_1) = g(X_1, Y_1)H, \tag{2.12}$$

for any $X_1,\, Y_1 \in \Gamma(TN)$.
A submanifold N is said to be totally geodesic if $h(X_1, Y_1) = 0$ and N is said to be minimal if $H = 0$.
Additionally, we define [22]

$$h^r_{ij} = g(h(e_i, e_j), e_r), \quad i, j = 1, 2, .., n, \quad r = n + 1, ..., 2m + 1,$$

and

$$\|h\|^2 = \sum_{i,j=1}^{n} g(h(e_i, e_j), h(e_i, e_j)).$$

For a differential function f on an n-dimensional manifold N, the gradient ∇f of f is defined by [11]

$$g(\nabla f, X_1) = X_1 f, \tag{2.13}$$

for any $X_1 \in \Gamma(TN)$. Consequently, for an orthonormal frame $\{e_1, e_2, ..., e_n\}$, we have [22]

$$\|\nabla f\|^2 = \sum_{i=1}^{n} (e_i(f))^2 \tag{2.14}$$

For any $X_1 \in \Gamma(TN)$, we can express ϕX_1 as

$$\phi X_1 = TX_1 + NX_1, \tag{2.15}$$

where TX_1 and NX_1 represent the tangential and normal components of ϕX_1 respectively. Similarly, for any vector $X_2 \in \Gamma(T^{\perp}N)$, we have

$$\phi X_2 = tX_2 + nX_2,$$

where tX_2 and nX_2 denote the tangential and normal components of ϕX_2, respectively. Furthermore, using (2.3) and (2.15), we get that

$$g(TX_1, Y_1) = -g(X_1, TY_1),$$

for any vector $X_1, Y_1 \in \Gamma(T^{\perp}N)$. For submanifolds tangent to the structure vector field ξ, there are various classes of submanifolds. We highlight the following [6]:

(1) A submanifold N tangent to ξ is called an invariant submanifold if ϕ preserves every tangent space of N, that is, $\phi(T_pN) \subseteq T_pN$, for every $p \in N$.

(2) A submanifold N tangent to ξ is termed an anti-invariant submanifold if ϕ maps every tangent space of N into the normal space, that is, $\phi(T_pN) \subseteq T^{\perp}_pN$, for every $p \in N$.

(3) A submanifold N of an almost contact manifold $\tilde{M}$, tangent to ξ, is referred to as a contact CR-submanifold if there exists a differential distribution D on N such that its orthogonal complementary distribution $D^{\perp}$ is anti-invariant. Specifically, the following conditions must be satisfied:
   (i) $TN = D \oplus D^{\perp} \oplus \langle\xi\rangle$,
   (ii) D is an invariant distribution, i.e., $\phi D \subseteq TN$,

(iii) $D^{\perp}$ is an anti-invariant distribution, i.e., $\phi D^{\perp} \subseteq T^{\perp}N$.

(4) ([15], [10]) A submanifold N of an almost contact manifold $\tilde{M}$ is said to be a slant submanifold if for every point $x \in N$ and any vector $X_1 \in T_xN$ the Wirtinger angle, defined as the angle between ϕX_1 and T_xN, remains constant and is denoted by $\theta \in [0, \frac{\pi}{2}]$. In this case, θ is referred to as the slant angle of N in $\tilde{M}$.

The invariant and anti-invariant submanifolds are special cases of slant submanifolds, where $\theta = 0$ and $\theta = \frac{\pi}{2}$, respectively. A slant submanifold that is neither invariant nor anti-invariant is referred to as a proper slant submanifold.

The following provides a useful characterization of slant submanifolds in almost contact manifold.

Theorem 2.1. [10] *Let N be a submanifold of a almost contact metric manifold $\tilde{M}$ with $\xi \in \Gamma(TN)$. Then, N is a slant submanifold if and only there exists a constant $\lambda \in [0, 1]$ such that*

$$T^2 = \lambda(-I + \eta \otimes \xi). \tag{2.16}$$

Furthermore, if θ is the slant angle of N, then $\lambda = cos^2\theta$.

Thus, the following are the consequences of the above theorem:

$$g(TX_1, TY_1) = cos^2\theta(g(X_1, Y_1) - \eta(X_1)\eta(Y_1)), \tag{2.17}$$

$$g(NX_1, NY_1) = sin^2\theta(g(X_1, Y_1) - \eta(X_1)\eta(Y_1)), \tag{2.18}$$

for any $X_1, X_2 \in \Gamma(TN)$.

Definition 2.2. [13] A CR-slant warped product $N = (N_T \times N_{\perp}) \times_f N_{\theta}$ is defined as $D_1 \oplus D_2$-mixed totally geodesic if $h(D_1, D_2) = 0$, where D_1, D_2 are distributions belonging to $\{D_T, D_{\perp}, D_{\theta}\}$.

Definition 2.3. [13] A submanifold N of an almost contact metric manifold $\tilde{M}$, tangent to ξ, is called a CR-slant warped product if it can be expressed as a warped product of the form $N = B \times_f N_{\theta}$, where the fiber N_{θ} is a proper slant and the base $B = N_T \times N_{\perp}$ is the CR-product of invariant and anti-invariant submanifolds of $\tilde{M}$.

The tangent bundle of the CR-slant warped product submanifold is decomposed as follows:

$$TN = D_T \oplus D_{\perp} \oplus D_{\theta} \oplus \langle \xi \rangle \,, \tag{2.19}$$

where D_T is an invariant distribution, $D_{\perp}$ is an anti-invariant distribution and D_{θ} is a proper slant distribution and $\langle \xi \rangle$ is the 1-dimensional distribution spanned by the structure vector field ξ.

Furthermore, if ν is the ϕ-invariant subspace of the normal bundle $T^{\perp}N$, then for a CR-slant warped product submanifold, the normal bundle $T^{\perp}N$ can be decomposed as follows:

$$T^{\perp}N = \phi D^{\perp} \oplus FD_{\theta} \oplus \nu.$$

3. CR-SLANT WARPED PRODUCT SUBMANIFOLDS OF A NEARLY KENMOTSU MANIFOLD

In this section, we examine CR-slant warped product submanifold of a nearly Kenmotsu manifold $\tilde{M}$ of the form $N = B \times_f N_{\theta}$, where $B = N_T \times N_{\perp}$, is a CR-product of invariant and anti-invairant submanifolds of $\tilde{M}$, N_θ is a slant submanifold and f denotes the warping function. We derive an inequality for this type of submanifold. Throughout this paper, we denote the corresponding tangent spaces of N_T, $N_\perp$ and N_θ by D_T, $D_\perp$ and D_θ, respectively. First, we present the following results for future reference.

Lemma 3.1. *Let $N = B \times_f N_\theta$ be a CR-slant warped product submanifold of a nearly Kenmotsu manifold $\tilde{M}$ where $B = N_T \times N_\perp$ is a CR-product submanifold tangent to ξ and N_θ is a proper slant submanifold of $\tilde{M}$. Then, for any $X_1, Y_1 \in \Gamma(TN_T)$, $X_2, Y_2 \in \Gamma(TN_\perp)$ and $X_3 \in \Gamma(TN_\theta)$, we have*

(i) $\xi(lnf) = 1$,

(ii) $2g(h(X_2, Y_2), NX_3) = g(h(X_2, X_3), \phi Y_2) + g(h(Y_2, X_3), \phi X_2)$,

(iii) $g(h(X_1, X_2), NX_3) = \frac{1}{2} g(h(X_1, X_3), \phi X_2)$,

(iv) $g(h(X_1, Y_1), NX_3) = 0$,

for any $X_1, Y_1 \in \Gamma(TN_T)$, $X_2, Y_2 \in \Gamma(TN_\perp)$ and $X_3 \in \Gamma(TN_\theta)$.

Proof. To prove part (i), we take, $X_3 \in \Gamma(TN_\theta)$ and note that ξ is tangent to B. Using equations (2.6) and (2.7), we derive the following:

$$\tilde{\nabla}_{X_3} \phi\xi - \phi\tilde{\nabla}_{X_3}\xi + \tilde{\nabla}_\xi \phi X_3 - \phi\tilde{\nabla}_\xi X_3 = -\phi X_3. \tag{3.1}$$

By applying (2.1), (2.8), and (1.1) in (3.1), we obtain

$$-2\phi h(X_3, \xi) - \xi(lnf)\phi X_3 = -\phi X_3. \tag{3.2}$$

Contracting (3.2) with ϕX_3, we obtain

$$\xi(lnf) = 1.$$

Now, we proceed to prove part (ii). By utilizing (2.8), (2.15), (1.1), (2.7), (2.9), and the orthogonality of the distributions, we derive:

$$g(h(X_2, Y_2), FX_3) = g((\tilde{\nabla}_{X_2}\phi)Y_2, X_3) + g(A_{\phi Y_2} X_2, X_3), \tag{3.3}$$

for $X_2, Y_2 \in \Gamma(TN_\perp)$, $X_3 \in \Gamma(TN_\theta)$.
By interchanging X_2 and Y_2, we obtain:

$$g(h(X_2, Y_2), FX_3) = g((\tilde{\nabla}_{Y_2}\phi)X_2, X_3) + g(A_{\phi X_2} Y_2, X_3). \tag{3.4}$$

From (3.3) and (3.4) along with (2.6) and (2.10), we derive part (ii).
Similarly, by using (2.8), (2.15), (1.1), (2.7), (2.9), and the orthogonality of distributions, we derive:

$$g(h(X_1, X_2), FX_3) = g((\tilde{\nabla}_{X_1}\phi)X_2, X_3) + g(A_{\phi X_2} X_1, X_3). \tag{3.5}$$

On the other hand, we also obtain

$$g(h(X_1, X_2), FX_3) = g((\tilde{\nabla}_{X_2}\phi)X_1, X_3) - \phi X_1(lnf)g(X_2, X_3) - X_1(lnf)g(X_2, TX_3). \tag{3.6}$$

By the orthogonality of the distributions, we obtain:

$$g(h(X_1, X_2), FX_3) = g((\tilde{\nabla}_{X_2}\phi)X_1, X_3). \tag{3.7}$$

Part (iii) follows from (3.5) and (3.7), along with (2.6) and (2.10).
To prove part (iv), let $X_1, Y_1 \in \Gamma(TN_T)$ and $X_3 \in \Gamma(TN_\theta)$. By utilizing (2.8), (1.1), (2.15),

(2.3), and (2.7), we find that:

$$\begin{aligned} g(h(X_1,Y_1),NX_3) &= g(\tilde{\nabla}_{X_1}Y_1,NX_3)-g(\nabla_{X_1}Y_1,NX_3)\\ &= g(\tilde{\nabla}_{X_1}Y_1,\phi X_3-TX_3)\\ &= -g(\phi\tilde{\nabla}_{X_1}Y_1,X_3)-g(\nabla_{X_1}Y_1,TX_3)\\ &= g((\tilde{\nabla}_{X_1}\phi)Y_1,X_3)-g(\tilde{\nabla}_{X_1}\phi Y_1,X_3)\\ &= g((\tilde{\nabla}_{X_1}\phi)Y_1,X_3). \end{aligned} \tag{3.8}$$

By interchanging X_1 and Y_1 in (3.8), we obtain:

$$g(h(X_1,Y_1),NX_3)=g((\tilde{\nabla}_{Y_1}\phi)X_1,X_3). \tag{3.9}$$

From (3.8) and (3.9), we derive

$$2g(h(X_1,Y_1),NX_3)=g((\tilde{\nabla}_{X_1}\phi)Y_1+(\tilde{\nabla}_{Y_1}\phi)X_1,X_3).$$

By substituting (2.6) into the above equation, we obtain part (iv). Therefore, the proof is complete. □

Lemma 3.2. *Let $N=B\times_f N_\theta$ be a CR-slant warped product submanifold of a nearly Kenmotsu manifold $\tilde{M}$ where $B=N_T\times N_\perp$ is a CR-product submanifold tangent to ξ and N_θ is a proper slant submanifold of $\tilde{M}$. Then, we have*

$$g(h(X_3,Y_3),\phi X_2)=\frac{1}{3}X_2(lnf)g(X_3,TY_3)+g(h(X_2,X_3),NY_3), \tag{3.10}$$

for any $X_2\in\Gamma(TN_\perp)$ and $X_3,Y_3\in\Gamma(TN_\theta)$.

Proof. Let $X_2\in\Gamma(TN_\perp)$, and $X_3,Y_3\in\Gamma(TN_\theta)$. Utilizing equations (2.8), (2.15), (1.1), (2.7), (2.9), along with the orthogonality of distributions, we derive the following result:

$$g(h(X_2,X_3),NY_3)=g((\tilde{\nabla}_{X_3}\phi)X_2,Y_3)+g(A_{\phi X_2}X_3,Y_3)-X_2(lnf)g(X_3,TY_3). \tag{3.11}$$

On the other hand, by utilizing equations (2.8), (2.15), (1.1), (2.7), (2.9) and applying the orthogonality of distributions, we derive:

$$g(h(X_2,X_3),NY_3)=g((\tilde{\nabla}_{X_2}\phi)X_3,Y_3)+g(A_{NX_3}X_2,Y_3). \tag{3.12}$$

Using equations (3.11) and (3.12), along with (2.6) and (2.10), we arrive at:

$$2g(h(X_2,X_3),NY_3)=g(h(X_3,Y_3),\phi X_2)+g(h(X_2,Y_3),NX_3)-X_2(lnf)g(X_3,TY_3). \tag{3.13}$$

Applying the polarization identity in (3.13), we obtain:

$$2g(h(X_2,Y_3),NX_3)=g(h(X_3,Y_3),\phi X_2)+g(h(X_2,X_3),NY_3)-X_2(lnf)g(Y_3,TX_3). \tag{3.14}$$

The desired result then follows from equations (3.13) and (3.14). □

From Lemma (3.2), the following equations can be deduced: by replacing X_3 by TX_3 (and Y_3 by TY_3) in (3.10) and utilizing (2.16) and (2.17), we obtain, respectively:

$$g(h(TX_3,Y_3),\phi X_2)=\frac{1}{3}X_2(lnf)cos^2\theta g(X_3,Y_3)+g(h(X_2,TX_3),NY_3). \tag{3.15}$$

$$g(h(X_3,TY_3),\phi X_2)=-\frac{1}{3}X_2(lnf)cos^2\theta g(X_3,Y_3)+g(h(X_2,X_3),NTY_3). \tag{3.16}$$

Additionally, by replacing X_3 by TX_3 in (3.16) and applying (2.17), we get

$$g(h(TX_3,TY_3),\phi X_2)=-\frac{1}{3}X_2(lnf)cos^2\theta g(TX_3,Y_3)+g(h(X_2,TX_3),NTY_3). \tag{3.17}$$

From (3.15) and (3.16), it can be observed that:

$$g(h(TX_3,Y_3),\phi X_2)+g(h(X_3,TY_3),\phi X_2)=g(h(X_2,TX_3),NY_3)+g(h(X_2,X_3),NTY_3). \tag{3.18}$$

Lemma 3.3. *Let $N=B\times_f N_\theta$ be a CR-slant warped product submanifold of a nearly Kenmotsu manifold $\tilde{M}$ where $B=N_T\times N_\perp$ is a CR-product submanifold tangent to ξ and N_θ is a proper slant submanifold of $\tilde{M}$. Then, we have*

$$g(h(X_1,X_3),NY_3)=\frac{1}{3}(X_1(lnf)-\eta(X_1))g(TX_3,Y_3)-\phi X_1(lnf)g(X_3,Y_3), \tag{3.19}$$

for any $X_1\in\Gamma(TN_T)$ and $X_3,Y_3\in\Gamma(TN_\theta)$.

Proof. Let $X_1\in\Gamma(TN_T)$ and $X_3,Y_3\in\Gamma(TN_\theta)$. By using equations (2.8), (2.15), (1.1), (2.7), (2.9) and applying the orthogonality of distributions, we derive:

$$g(h(X_1,X_3),NY_3)=g((\tilde{\nabla}_{X_1}\phi)X_3,Y_3)+g(A_{NX_3}X_1,Y_3). \tag{3.20}$$

In a similar manner, by using (2.8), (2.15), (1.1), (2.7), (2.9) and applying the orthogonality of distributions, we derive:

$$g(h(X_1,X_3),NY_3)=g(h(\tilde{\nabla}_{X_3}\phi)X_1,Y_3)-\phi X_1(lnf)g(X_3,Y_3)-X_1(lnf)g(X_3,TY_3). \tag{3.21}$$

Then, with the help of (2.6) and (2.10), equations (3.20) and (3.21) give

$$\begin{aligned}2g(h(X_1,X_3),NY_3)=&g(h(X_1,Y_3),NX_3)-\eta(X_1)g(\phi X_3,Y_3)-\phi X_1(lnf)g(X_3,Y_3)\\&-X_1(lnf)g(X_3,TY_3).\end{aligned} \tag{3.22}$$

Using the polarization identity, we obtain:

$$\begin{aligned}2g(h(X_1,Y_3),NX_3)=&g(h(X_1,X_3),NY_3)-\eta(X_1)g(\phi Y_3,X_3)-\phi X_1(lnf)g(X_3,Y_3)\\&-X_1(lnf)g(Y_3,TX_3).\end{aligned} \tag{3.23}$$

Therefore, equation (3.19) follows from (3.22) and (3.23), thus proving the lemma. □

From Lemma (3.3) and using (2.17), we derive the following equations. In particular, by replacing X_1 with ϕX_1, X_3 with TX_3 and Y_3 with TY_3 in (3.19), we get

$$g(h(\phi X_1,X_3),NY_3)=(X_1(lnf)-\eta(X_1))g(X_3,Y_3)+\frac{1}{3}\phi X_1(lnf)g(TX_3,Y_3), \tag{3.24}$$

$$g(h(X_1,TX_3),NY_3)=-\frac{1}{3}(X_1(lnf)-\eta(X_1))cos^2\theta g(X_3,Y_3)-\phi X_1(lnf)g(TX_3,Y_3), \tag{3.25}$$

$$g(h(X_1,X_3),NTY_3)=\frac{1}{3}(X_1(lnf)-\eta(X_1))cos^2\theta g(X_3,Y_3)-\phi X_1(lnf)g(X_3,TY_3). \tag{3.26}$$

Similarly, by replacing X_3 with TX_3 (and Y_3 with TY_3) in (3.24), we obtain

$$g(h(\phi X_1,TX_3),NY_3)=(X_1(lnf)-\eta(X_1))g(TX_3,Y_3)-\frac{1}{3}\phi X_1(lnf)cos^2\theta g(X_3,Y_3), \tag{3.27}$$

$$g(h(\phi X_1,X_3),NTY_3)=(X_1(lnf)-\eta(X_1))g(X_3,TY_3)+\frac{1}{3}\phi X_1(lnf)cos^2\theta g(X_3,Y_3). \tag{3.28}$$

From (3.27) and by replacing Y_3 with TY_3, we have

$$g(h(\phi X_1, TX_3), NTY_3) = (X_1(lnf) - \eta(X_1))cos^2\theta g(X_3, Y_3) - \frac{1}{3}\phi X_1(lnf)cos^2\theta g(X_3, TY_3). \tag{3.29}$$

From (3.26) and by replacing X_3 with TX_3, we obtain

$$g(h(X_1, TX_3), NTY_3) = \frac{1}{3}(X_1(lnf) - \eta(X_1))cos^2\theta g(TX_3, Y_3) - \phi X_1(lnf)cos^2\theta g(X_3, Y_3). \tag{3.30}$$

Clearly, from (3.25) and (3.26), and similarly from (3.27) and (3.28), we derive the following, respectively:

$$g(h(X_1, TX_3), NY_3) + g(h(X_1, X_3), NTY_3) = 0, \tag{3.31}$$

$$g(h(\phi X_1, TX_3), NY_3) + g(h(\phi X_1, X_3), NTY_3) = 0. \tag{3.32}$$

Substituting $X_1 = \xi$ in (3.19) and (3.25), we observed that:

$$g(h(\xi, X_3), NY_3) = 0, \tag{3.33}$$

and

$$g(h(\xi, TX_3), NY_3) = 0. \tag{3.34}$$

Theorem 3.4. *Let $N = B \times_f N_\theta$ be a CR-slant warped product submanifold of a nearly Kenmotsu manifold $\tilde{M}$ where $B = N_T \times N_\perp$ is a CR-product submanifold tangent to ξ and N_θ is a proper slant submanifold of $\tilde{M}$. If N is $D_T \oplus D_\theta$-mixed totally geodesic then, f depends only on $N_\perp$, i.e., f is constant along N_T.*

Proof. Given that N is a $D_T \oplus D_\theta$-mixed totally geodesic CR-slant warped product submanifold. Then equations (3.19) and (3.27) reduce to:

$$\frac{1}{3}(X_1(lnf) - \eta(X_1))g(TX_3, Y_3) - \phi X_1(lnf)g(X_3, Y_3) = 0, \tag{3.35}$$

and

$$(X_1(lnf) - \eta(X_1))g(TX_3, Y_3) - \frac{1}{3}\phi X_1(lnf)cos^2\theta g(X_3, Y_3) = 0. \tag{3.36}$$

From equations (3.35) and (3.36), we obtain

$$(cos^2\theta - 9)\phi X_1(lnf)g(X_3, Y_3) = 0, \tag{3.37}$$

for any $X_1 \in \Gamma(D_T)$ and $X_3, Y_3 \in \Gamma(D_\theta)$. Since g is a Riemannian metric, we either get $cos\theta = \pm 3$ which is not possible, or $\phi X_1(lnf) = 0$, implying that f is constant along N_T. This concludes the proof of the theorem. □

Theorem 3.5. *Let $N = B \times_f N_\theta$ be a $D_\perp \oplus D_\theta$-mixed totally geodesic CR-slant warped product submanifold of a nearly Kenmotsu manifold $\tilde{M}$ such that ξ is tangent to N_T, where $B = N_T \times N_\perp$ is a CR-product submanifold and N_θ is a slant submanifold of $\tilde{M}$. Then, the second fundamental form h of N satisfies the following inequality:*

$$\|h\|^2 \geq 4l\left(csc^2\theta + \frac{1}{9}cot^2\theta\right)\left[\|\nabla_T(lnf)\|^2 - 1\right] + \frac{2l}{9}cos^2\theta\|\nabla_\perp(lnf)\|^2, \tag{3.38}$$

where $\nabla_T(lnf)$ and $\nabla_\perp(lnf)$ represents the gradient of the function lnf in the directions of N_T and $N_\perp$, respectively, and $l = \frac{1}{2}dimN_\theta$.

If the equality holds identically in (3.38), then B is totally geodesic submanifold of $\tilde{M}$ and N_θ is a totally umbilical submanifold of $\tilde{M}$, respectively. Furthermore, N is a $D_T \oplus D_\perp$-mixed totally geodesic submanifold of $\tilde{M}$ but not $D_T \oplus D_\theta$-mixed totally geodesic. Therefore, N is not minimal in $\tilde{M}$.

Proof. Let $\tilde{M}$ be a $(2m+1)$-dimensional nearly Kenmotsu manifold and let $N = B \times_f N_\theta$ be an n-dimensional CR-slant warped product submanifold, where $B = N_T \times N_\perp$ is a CR-product submanifold tangent to ξ.

Let us consider $dimN_T = 2s+1$, $dimN_\perp = t$ and $dimN_\theta = 2l$. We define the following local orthonormal frames: on N_T, $\{e_1, e_2, ..., e_s, e_{s+1} = \phi e_1, ..., e_{2s} = \phi e_s, e_{2s+1} = \xi\}$; on $N_\perp$, $\{e_{2s+2} = \tilde{e}_1, ..., e_{2s+t+1} = \tilde{e}_t\}$; and on N_θ, $\{e_1^* = e_{2s+t+2}, ..., e_l^* = e_{2s+t+l+1}, e_{l+1}^* = sec\theta TE_1^*, ..., e_{2l}^* = sec\theta Te_l^*\}$.

The orthonormal frames in the normal bundle $T^\perp N$ of $\phi D^\perp$ are $\{e_{n+1} = \hat{e_1} = \phi\tilde{e}_1, ..., e_{n+t} = \hat{e}_t = \phi\tilde{e}_t\}$, the orthonormal basis ND_θ is $\{e_{n+t+1} = \hat{e}_{t+1} = csc\theta Ne_1^*, ..., e_{n+t+l} = \hat{e}_{t+l} = csc\theta Ne_l^*, ..., e_{n+t+l+1} = \hat{e}_{t+l+1} = csc\theta sec\theta NTe_1^*, ..., e_{n+t+2l} = \hat{e}_{t+2l} = csc\theta sec\theta NTe_l^*\}$, and the invariant normal subbundle ν is $\{e_{n+t+2l+1}, ..., e_{2m+1}\}$, respectively.

According to the definition of h, we have

$$\|h\|^2 = \sum_{i,j=1}^{n} g(h(e_i, e_j), h(e_i, e_j)) = \sum_{r=n+1}^{2m+1} \sum_{i,j=1}^{n} g(h(e_i, e_j), e_r)^2.$$

For the given frames, the above equation can be written as:

$$\begin{aligned}\|h\|^2 = {} & \sum_{r=n+1}^{n+t} \sum_{i,j=1}^{n} g(h(e_i, e_j), e_r)^2 + \sum_{r=n+t+1}^{n+t+2l} \sum_{i,j=1}^{n} g(h(e_i, e_j), e_r)^2 \\ & + \sum_{r=n+t+2l+1}^{2m+1} \sum_{i,j=1}^{n} g(h(e_i, e_j), e_r)^2.\end{aligned} \tag{3.39}$$

The first term on the right-hand side of (3.39) represents the $\phi D^\perp$-component, the second term corresponds to the FD_θ-component, and the third term represents the ν-component. By omitting the third term and decomposing the first two terms on the right-hand side of (3.39) with respect to the orthonormal frame fields of D_T, $D_\perp$, and D_θ, we obtain

$$\begin{aligned}\|h\|^2 \geq {} & \sum_{r=1}^{t} \sum_{i,j=1}^{2s+1} g(h(e_i, e_j), \phi\tilde{e}_r)^2 + 2\sum_{r=1}^{t} \sum_{i=1}^{2s+1} \sum_{j=1}^{t} g(h(e_i, \tilde{e}_j), \phi\tilde{e}_r)^2 \\ & + 2\sum_{r=1}^{t} \sum_{i=1}^{2s+1} \sum_{j=1}^{2l} g(h(e_i, e_j^*), \phi\tilde{e}_r)^2 + \sum_{r=1}^{t} \sum_{i,j=1}^{t} g(h(\tilde{e}_i, \tilde{e}_j), \phi\tilde{e}_r)^2 \\ & + 2\sum_{r=1}^{t} \sum_{i=1}^{t} \sum_{j=1}^{2l} g(h(\tilde{e}_i, e_j^*), \phi\tilde{e}_r)^2 + \sum_{r=1}^{t} \sum_{i,j=1}^{2l} g(h(e_i^*, e_j^*), \phi\tilde{e}_r)^2 \\ & + \sum_{r=t+1}^{t+2l} \sum_{i,j=1}^{2s+1} g(h(e_i, e_j), \hat{e}_r)^2 + 2\sum_{r=t+1}^{t+2l} \sum_{i=1}^{2s+1} \sum_{j=1}^{t} g(h(e_i, \tilde{e}_j), \hat{e}_r)^2\end{aligned}$$

(3.40)

$$+2\sum_{r=t+1}^{t+2l}\sum_{i=1}^{2s+1}\sum_{j=1}^{2l}g(h(e_i,e_j^*),\hat{e}_r)^2+\sum_{r=t+1}^{t+2l}\sum_{i,j=1}^{t}g(h(\tilde{e}_i,\tilde{e}_j),\hat{e}_r)^2$$
$$+2\sum_{r=t+1}^{t+2l}\sum_{i=1}^{t}\sum_{j=1}^{2l}g(h(\tilde{e}_i,e_j^*),\hat{e}_r)^2+\sum_{r=t+1}^{t+2l}\sum_{i,j=1}^{2l}g(h(e_i^*,e_j^*),\hat{e}_r)^2.$$

The fifth and eleventh terms on the right-hand side of (3.40) vanish identically, as N is $D_\perp \oplus D_\theta$-mixed geodesic. Similarly, the seventh and tenth terms vanish identically by applying Lemma (3.1). Additionally, we could not find the relations for warped products of the first, second, fourth and twelth terms in (3.40). As a result, we will keep these positive terms but consider them in the equality case. We now proceed to compute the third, sixth, eighth and ninth terms on the right-hand side of (3.40), that is

$$\begin{aligned}\|h\|^2 \geq{}& 2\sum_{r=1}^{t}\sum_{i=1}^{2s+1}\sum_{j=1}^{2l}g(h(e_i,e_j^*),\phi\tilde{e}_r)^2+\sum_{r=1}^{t}\sum_{i,j=1}^{2l}g(h(e_i^*,e_j^*),\phi\tilde{e}_r)^2\\ &+2\sum_{r=t+1}^{t+2l}\sum_{i=1}^{2s+1}\sum_{j=1}^{2l}g(h(e_i,e_j^*),\hat{e}_r)^2+2\sum_{r=t+1}^{t+2l}\sum_{i=1}^{2s+1}\sum_{j=1}^{t}g(h(e_i,\tilde{e}_j),\hat{e}_r)^2.\end{aligned}\tag{3.41}$$

By applying part (iii) of Lemma (3.1) to equation (3.41), we obtain

$$\begin{aligned}h\|^2 \geq{}& \sum_{r=1}^{t}\sum_{i,j=1}^{2l}g(h(e_i^*,e_j^*),\phi\tilde{e}_r)^2+2\sum_{r=t+1}^{t+2l}\sum_{i=1}^{2s+1}\sum_{j=1}^{2l}g(h(e_i,e_j^*),\hat{e}_r)^2\\ &+10\sum_{r=t+1}^{t+2l}\sum_{i=1}^{2s+1}\sum_{j=1}^{t}g(h(e_i,\tilde{e}_j),\hat{e}_r)^2.\end{aligned}\tag{3.42}$$

Excluding the third term, we decompose the first and second terms on the right-hand side of (3.42) as follows:

$$\begin{aligned}\sum_{r=1}^{t}\sum_{i,j=1}^{2l}g(h(e_i^*,e_j^*),\phi\tilde{e}_r)^2={}&\sum_{r=1}^{t}\sum_{i,j=1}^{l}[g(h(e_i^*,e_j^*),\phi\tilde{e}_r)^2+g(h(sec\theta Te_i^*,e_j^*),\phi\tilde{e}_r)^2\\ &+g(h(e_i^*,sec\theta Te_j^*),\phi\tilde{e}_r)^2+g(h(sec\theta Te_i^*,sec\theta Te_j^*),\phi\tilde{e}_r)^2].\end{aligned}\tag{3.43}$$

Given that N is a $D_\perp \oplus D_\theta$-mixed geodesic, and by equation (3.18), the second and third terms in (3.43) vanish, this leads to:

$$\sum_{r=1}^{t}\sum_{i,j=1}^{2l}g(h(e_i^*,e_j^*),\phi\tilde{e}_r)^2=\sum_{r=1}^{t}\sum_{i,j=1}^{l}\left[g(h(e_i^*,e_j^*),\phi\tilde{e}_r)^2+sec^4\theta g(h(Te_i^*,Te_j^*),\phi\tilde{e}_r)^2\right].\tag{3.44}$$

By substituting equations (3.10), (3.17), and (2.14) into (3.44), we derive

$$\begin{aligned}\sum_{r=1}^{t}\sum_{i,j=1}^{2l}g(h(e_i^*,e_j^*),\phi\tilde{e}_r)^2={}&\frac{2l}{9}cos^2\theta\sum_{r=1}^{t}(\tilde{e}_r(lnf))^2\\ ={}&\frac{2l}{9}cos^2\theta\|\nabla_\perp(lnf)\|^2.\end{aligned}\tag{3.45}$$

Next, we decompose the second term on the right-hand side of (3.42), and by using equations (3.31) and (3.32), we find that:

$$\begin{aligned} 2\sum_{r=t+1}^{t+2l}\sum_{i=1}^{2s+1}\sum_{j=1}^{2l} g(h(e_i,e_j^*),\hat{e}_r)^2 = 2\sum_{i=1}^{s}\sum_{r,j=1}^{l} &[g(h(e_i,e_j^*), csc\theta N e_r^*)^2 \\ &+ g(h(e_i, sec\theta T e_j^*), csc\theta sec\theta NT e_r^*)^2 \\ &+ g(h(\phi e_i, e_j^*), csc\theta N e_r^*)^2 \\ &+ g(h(\phi e_i, sec\theta T e_j^*), csc\theta sec\theta NT e_r^*)^2 \\ &+ 2\sum_{r=t+1}^{t+2l}\sum_{j=1}^{2l} g(h(\xi, e_j^*),\hat{e}_r)^2. \end{aligned} \tag{3.46}$$

By employing (3.19), (3.24), (3.29), (3.30) in (3.46) and noting that $\eta(e_i) = 0$ for any $i \in \{1,2,...,2s\}$, we have

$$\begin{aligned} 2\sum_{r=t+1}^{t+2l}\sum_{i=1}^{2s+1}\sum_{j=1}^{2l} g(h(e_i,e_j^*),\hat{e}_r)^2 =& 4l\left(csc^2\theta + \frac{1}{9}cot^2\theta\right)\left[\sum_{i=1}^{2s+1}(e_i(lnf))^2 - (\xi(lnf))^2\right] \\ &+ 2\sum_{r=t+1}^{t+2l}\sum_{j=1}^{2l} g(h(\xi,e_j^*),\hat{e}_r)^2. \end{aligned} \tag{3.47}$$

we decompose the last term on the right side of (3.47) and by using (3.31), yields

$$2\sum_{r=t+1}^{t+2l}\sum_{j=1}^{2l} g(h(\xi,e_j^*),\hat{e}_r)^2 = 2\sum_{r=1}^{l}\sum_{j=1}^{l}\left[g(h(\xi,e_j^*), csc\theta N e_r^*)^2 + g(h(\xi, sec\theta T e_j^*), csc\theta sec\theta NT e_r^*)^2\right].$$

By applying equations (3.33) and (3.34) to the previous equation, we obtain

$$2\sum_{r=t+1}^{t+2l}\sum_{j=1}^{2l} g(h(\xi,e_j^*),\hat{e}_r)^2 = 0. \tag{3.48}$$

By employing equation (2.14), part (i) of Lemma (3.1), and equation (3.48), we can express equation (3.47) as:

$$2\sum_{r=t+1}^{t+2l}\sum_{i=1}^{2s+1}\sum_{j=1}^{2l} g(h(e_i,e_j^*),\hat{e}_r)^2 = 4l\left(csc^2\theta + \frac{1}{9}cot^2\theta\right)\left[\|\nabla_T(lnf)\|^2 - 1\right]. \tag{3.49}$$

Therefore, by substituting (3.45) and (3.49) in (3.42), we obtain (3.38), thereby proving inequality.

If equality holds in (3.38), then from the leaving third term on the right-hand side of (3.39), we have

$$h(X_1, X_2) \perp \nu, \tag{3.50}$$

for any $X_1, X_2 \in \Gamma(TN)$. In addition, from the leaving first term and considering the vanishing seventh term in (3.40), we find

$$h(D_T, D_T) \perp \phi D_\perp, \quad h(D_T, D_T) \perp F D_\theta. \tag{3.51}$$

Then, by using equations (3.50) and (3.51), we obtain

$$h(D_T, D_T) = \{0\}. \tag{3.52}$$

Similarly, from the leaving fourth term and vanishing tenth term in (3.40), we observe that

$$h(D_\perp, D_\perp) \perp \phi D_\perp, \quad h(D_\perp, D_\perp) \perp FD_\theta. \tag{3.53}$$

Thus, (3.50) and (3.53) yield:

$$h(D_\perp, D_\perp) = \{0\}. \tag{3.54}$$

Furthermore, from the leaving second term in (3.40) and leaving third term in (3.42), we get

$$h(D_T, D_\perp) \perp \phi D_\perp, \quad h(D_T, D_\perp) \perp FD_\theta. \tag{3.55}$$

Then, from (3.50) and (3.55), we obtain

$$h(D_T, D_\perp) = \{0\}. \tag{3.56}$$

Since N is $D_\perp \oplus D_\theta$-mixed geodesic in $\tilde{M}$, we derive

$$h(D_\perp, D_\theta) = \{0\}. \tag{3.57}$$

On the other hand, from the leaving twelth term in (3.40) along with (3.50), we obtain

$$h(D_\theta, D_\theta) \subset \phi D_\perp. \tag{3.58}$$

Since B is totally geodesic in N ([7, 11]), this fact, combined with (3.52) and (3.54), implies that B is totally geodesic in $\tilde{M}$. Also, since N_θ is totally umbilical in N ([7, 11]), this fact, together with (3.58), leads to the conclusion that N_θ is totally umbilical in $\tilde{M}$. Furthermore, from (3.56), we can deduce that N is a $D_T \oplus D_\perp$-mixed geodesic submanifold of $\tilde{M}$. However, from Theorem (3.4), N can never be a $D_T \oplus D_\theta$-mixed geodesic. Therefore, the theorem is proved. □

If the structure vector field ξ is tangent to $N_\perp$, then the following result holds.

Theorem 3.6. *Let $N = B \times_f N_\theta$ be a $D_\perp \oplus D_\theta$-mixed totally geodesic CR-slant warped product submanifold of a nearly Kenmotsu manifold $\tilde{M}$ such that ξ is tangent to $N_\perp$, where $B = N_T \times N_\perp$ is a CR-product submanifold and N_θ is a slant submanifold of $\tilde{M}$. Then, the second fundamental form h of N satisfies the following inequality:*

$$\|h\|^2 \geq 4l\left(csc^2\theta + \frac{1}{9}cot^2\theta\right)\|\nabla_T(lnf)\|^2 + \frac{2l}{9}cos^2\theta\left[\|\nabla_\perp(lnf)\|^2 - 1\right], \tag{3.59}$$

If the equality holds identically in (3.59), then B is totally geodesic submanifold of $\tilde{M}$ and N_θ is a totally umbilical submanifold of $\tilde{M}$, respectively. Furthermore, N is a $D_T \oplus D_\perp$-mixed totally geodesic submanifold of $\tilde{M}$ but not $D_T \oplus D_\theta$-mixed totally geodesic. Therefore, N is not minimal in $\tilde{M}$.

Proof. In this theorem, we consider $dimN_T = 2s$ and $dimN_\perp = t+1$. Thus, the orthonormal frames for D_T and $D_\perp \oplus \{\xi\}$ are given by $\{e_1, e_2, ..., e_s, e_{s+1} = \phi e_1, ..., e_{2s} = \phi e_s\}$ and $\{e_{2s+1} = \tilde{e}_1, ..., e_{2s+t} = \tilde{e}_t, \tilde{e}_{2s+t+1} = \xi\}$, respectively. The proof of this theorem follows in a similar manner to Theorem (3.5). □

4. APPLICATIONS

As a consequence of Theorem (3.5) and Theorem (3.6), several special cases of CR-slant warped product submanifolds can be derived.

Case (i). If dimN_T=0 in Theorem (3.6), then we have

Theorem 4.1. *(Theorem 4.1 [1]) Let $N = N_\perp \times_f N_\theta$ be a mixed totally geodesic proper pseudo-slant warped product submanifold of a nearly Kenmotsu manifold $\tilde{M}$. Then, the second fundamental form h of N satisfies the following condition:*

$$\|h\|^2 \geq \frac{2l}{9} cos^2\theta \left[\|\nabla_\perp(lnf)\|^2 - 1\right], \tag{4.1}$$

where $2l$=$dimN_\theta$ and $\nabla_\perp(lnf)$ is the gradient of lnf.
Moreover, if equality holds in (4.1), then $N_\perp$ becomes a totally geodesic submanifold of $\tilde{M}$ and N_θ is a totally umbilical submanifold of $\tilde{M}$. Furthermore, N is minimal in $\tilde{M}$.

Case (ii). If dim$N_\perp$=0 in Theorem (3.5), then we state the following Theorem.

Theorem 4.2. *Let $N = N_T \times_f N_\theta$ be a warped product semi-slant submanifold of a nearly Kenmotsu manifold $\tilde{M}$. Then, the second fundamental form h of N satisfies the following condition:*

$$\|h\|^2 \geq 4l \left(csc^2\theta + \frac{1}{9}cot^2\theta\right) \left[\|\nabla_T(lnf)\|^2 - 1\right], \tag{4.2}$$

where l=$\frac{1}{2}$dimN_θ and $\nabla_T(lnf)$ is the gradient of ln$f$.
Moreover, if equality holds in (4.2), then N_T is a totally geodesic submanifold of $\tilde{M}$ and N_θ is a totally umbilical submanifold of $\tilde{M}$. Furthermore, N is never a mixed totally geodesic submanifold and thus N is not minimal in $\tilde{M}$.
When $\alpha = 0$ and $\beta = 1$ in Theorem 4.1 [17], inequality (4.1) from Theorem 4.1 in [17] aligns with inequality (4.2). Therefore, Theorem (3.5) can be viewed as an extension of Theorem 4.1 [17] under certain conditions.

Case (iii). If dimN_θ=0 in Theorem (3.5), then we state the following Theorem.

Theorem 4.3. *Let $N = N_T \times_f N_\perp$ be a CR-warped product submanifold of a nearly Kenmotsu manifold $\tilde{M}$. Then, the second fundamental form h of N satisfies the following condition:*

$$\|h\|^2 \geq 2l \left[\|\nabla_T(lnf)\|^2 - 1\right], \tag{4.3}$$

where l=dim$N_\perp$ and $\nabla_T(lnf)$ is the gradient of ln$f$.

Moreover, if equality holds in (4.3), then N_T becomes totally geodesic submanifold of $\tilde{M}$ and N_θ is a totally umbilical submanifold of $\tilde{M}$. Furthermore, N is a minimal submanifold of $\tilde{M}$.
It is important to note that inequality (4.3) aligns with inequality (7) from Theorem 3.1 in [5], indicating that our derived Theorem (3.3) serves as a generalization of Theorem 3.1 from [5]. On another note, if we set $\alpha = 0$ and $\beta = 1$ in Theorem 4.1 of [16], the nearly trans Sasakian manifold reduces to a nearly Kenmotsu manifold. Thus, inequality (4.1) in Theorem 4.1 of [16] corresponds to inequality (4.3). It is clear that Theorem 4.1 from [16] is a special case of our Theorem (3.5).

5. CONCLUSIONS

This work investigates CR-slant warped product submanifolds of the form $B \times_f N_\theta$ in nearly Kenmotsu manifolds. We establish a sharp inequality for the squared norm of the second fundamental form in terms of the warping function, taking into account the position of the structure vector field with respect to the invariant and anti-invariant components of the CR-product. Furthermore, the equality cases are completely characterized, and several geometric consequences are derived, extending and generalizing earlier results on warped product submanifolds in contact metric geometry.

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