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Research Paper

FRACTIONAL ORDER NONLINEAR RANDOM DIFFERENTIAL EQUATION IN SEPARABLE BANACH ALGEBRA

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ABSTRACT

This paper investigates fractional-order differential equations with random operators in a separable Banach algebra. The main goal is to prove the existence of random solutions for these equations. We focus on an initial value problem involving fractional order differential equations with random operators. By transforming this initial value problem into an equivalent integral form and applying the fixed point theorem along with the Caratheodory condition, we formulate some hypothesis for the existence of random solutions. Additionally, we demonstrate that these random solutions possess local attractivity. This study provides a framework for understanding and solving existence of random solutions for fractional order differential equations with random operators. An illustrative example is also provided to verify and demonstrate the applicability of the main result.

1. **Introduction**

A useful tool for examining issues in the real world is the differential equation. The mean of the values obtained from specific experimental measurements is commonly used to represent the coefficients, parameters, and initial conditions when creating a differential equation. On the basis of probability distributions or laws, physical constants and parameters can be thought of as random variables. The study of fractional order differential and integral

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problems has recently taken up a significant part of nonlinear analysis. Applied science and engineering benefit from fractional differentiation[10]. In scientific domains like physics, chemistry, biology, economics, signal and image processing, feedback amplifiers, and electric circuits, fractional differential equations are frequently used in the mathematical modelling of systems and processes. Examples of these include viscoelasticity[9] [10]. Fractional Bacterial growth is described using fractional calculus[3]. Deterministic fixed point theorems are stochastically generalised by random fixed point theorems. The use of random fixed point theorems to demonstrate the presence of random solutions for differential and integral equations has made them more important in recent years [1][6][7]. See [5] [4] for a detailed explanation of random fixed point theorems of two and three operators. The paper [11] describes the existence of a mild solution to the Cauchy problem for impulsive semi-linear second-order differential inclusion in a Banach space. The paper [8] investigates the existence and controllability of second-order functional diferential equations with infnite delay, incorporating random operators.

In this, we investigates fractional-order differential equations with random operators in a separable Banach algebra. The existence and attractivity of solutions to fractional order random differential equations have been demonstrated via fixed point theory. The illustrative example confirms the applicability of the theoretical results and demonstrates the effectiveness of the proposed approach.

2. Statement of problem

The following fractional order nonlinear random differential equation (FONRDE) is taken into consideration:

$$D^{\varsigma}\left(\frac{x(t,\omega)}{\mathcal{J}(t,x(t,\omega),\omega)}\right)=\mathcal{K}\left(t,x(t,\omega),\omega\right)$$

$$x(0,\omega)=0 \tag{2.1}$$

for all $\omega \in \Omega,\ t \in \mathcal{R}_+$ and $x \in BM(\mathcal{R}_+,\mathcal{R})$,
Where $\mathcal{J}:\mathcal{R}_+\times\mathcal{R}\times\Omega\to\mathcal{R}-\{0\}$; $\mathcal{K}:\mathcal{R}_+\times\mathcal{R}\times\Omega\to\mathcal{R}$.
The fractional order nonlinear random differential equation FONRDE (2.1) can be solved to provide the function $x \in BM(\mathcal{R}_+,\mathcal{R})$, where BM is a bounded measurable function which is such that

i) The function $t \to \left(\frac{x}{\mathcal{J}(t,x(t,\omega),\omega)}\right)$ is bounded measurable for each $x \in \mathcal{R}$,
ii) x satisfies (2.1).

3. Preliminaries

The fundamental concepts, lemmas, and background information that are utilized throughout this article are reviewed in this part.

Definition 3.1. [6]: Given a measurable space $(\Omega,\mathcal{A})$ and a separable Banach space $\mathcal{S}$ with σ-algebra β_{s} , of every Borel subset of $\mathcal{S}$. For each Borel subset $\mathcal{B}$ of $\mathcal{S}$, a mapping $f:\Omega\to\mathcal{S}$ is said to be measurable if $f^{-1}(\mathcal{B})=\{\omega\in\Omega : f(\omega)\in\mathcal{B}\}\in\mathcal{A}$.

Definition 3.2. [12] : If $\Omega \to \mathcal{T}(\omega,s)$ is measurable for all $s\in\mathcal{S}$, then a mapping $\mathcal{T}:\Omega\times\mathcal{S}\to\mathcal{S}$ is a random operator, and we represent it as $\mathcal{T}(\omega)s=\mathcal{T}(\omega,s)$.

Definition 3.3. [8] : $\mathcal{T} : \Omega \times \mathcal{S} \to \mathcal{S}$ be a random operator, A random variable $\xi : \Omega \to \mathcal{S}$ is called random fixed point if $\mathcal{T}(\omega)\xi(\omega) = \xi(\omega)$ for every $\omega \in \Omega$.

Definition 3.4. A random operator $\mathcal{T} : \Omega \times \mathcal{S} \to \mathcal{S}$ is referred to be a continuous random operator iff $\lim_{n\to\infty} \|s_n - s\| = 0 \to \lim_{n\to\infty} \|\mathcal{T}(\omega)s_n - \mathcal{T}(\omega)s\| = 0$.

Definition 3.5. [5] : A random operator $\mathcal{T} : \Omega \times \mathcal{S} \to \mathcal{S}$ is said to be totally bounded if, for every $\omega \in \Omega$, $\mathcal{T}(\omega)(D)$ is a totally bounded subset of $\mathcal{S}$ for every bounded subset $\mathcal{D}$ of $\mathcal{S}$.

Definition 3.6. [5] : A random operator $\mathcal{T} : \Omega \times \mathcal{S} \to \mathcal{S}$ is referred to as a compact random operator, provided that $\overline{\mathcal{T}(\omega)(\mathcal{S})}$ is the closure of $\mathcal{T}(\omega)(\mathcal{S})$ and is a compact subset of $\mathcal{S}$.

Definition 3.7. [5]: In Banach space $\mathcal{S}$, a random operator $\mathcal{T}(\omega)$ is said to be completely continuous if it is both continuous and totally bounded on $\mathcal{S}$. Although the converse may not always be true, keep in mind that all compact operators are completely bounded.

Definition 3.8. [6]: Random solutions of random equations are said to be locally attractive on $\mathcal{R}_+$ if there is a closed ball $\bar{B}_a(x_0)$ in the space $S = BC(\mathcal{R}_+, \mathcal{R})$, with for some $x_0 \in S$ and for some real number $a > 0$, such that for arbitrary random solutions $x = x(t, \omega)$ and $y = y(t, \omega)$ of the random equation belonging to $\bar{B}_a(x_0) \cap \mathfrak{S}$, where $\mathfrak{S}$ is a non-empty subset of S, we have $\lim_{t\to\infty} \mid x(t, \omega) - y(t, \omega) \mid = 0$ for all $\omega \in \Omega$.

Definition 3.9. [2]: The Riemann-Liouville fractional integral of order $\zeta > 0$ of a continuous function $f \in L^1[0, T]$ is defined by

$$I^\zeta f(t) = \frac{1}{\Gamma(\zeta)} \int_0^t (t - s)^{(\zeta-1)} f(s)\, ds$$

Provided right-hand side is defined pointwise on $(0, \infty)$.

Definition 3.10. [2]: The Riemann-Liouville fractional derivative of order $\zeta > 0$ of a continuous function $g \in L^1[0, T]$ is defined by

$$D^\zeta g(t) = \frac{1}{\Gamma(n - \zeta)} \left(\frac{d}{dt}\right)^n \int_0^t (t - s)^{(n-\zeta-1)} g(s)\, ds$$

Where, $n - 1 < \zeta < n$, $n = [\zeta] + 1$ and $[\zeta]$ denotes the integer part of ζ, Γ denotes the Euler gamma function. The Riemann-Liouville fractional derivative of order ζ defined by $D^\zeta = \frac{d^\zeta}{dt^\zeta} = \frac{d}{dt} I^{1-\zeta}$.

Theorem 3.11. [7] *(**Arzela-Ascoli theorem**) Any sequence $\{g_n\}$ of functions in $C(J, \mathcal{R})$ that is uniformly bounded and equi-continuous has a convergent subsequence.*

Theorem 3.12. [7] *A metric space S is compact iff every sequence in S has convergent subsequence.*

Definition 3.13. [5]: A non-decreasing continuous function $\varphi : \Omega \times \mathcal{R}_+ \to \mathcal{R}_+$ exists such that for every $\omega \in \Omega$

$$\|\mathcal{T}(\omega)s_1 - \mathcal{T}(\omega)s_2\| \le \varphi_\omega(\|s_1 - s_2\|)$$

For all $s_1, s_2 \in \mathcal{S}$, where $\varphi_\omega(r) = \varphi(\omega, r)$ and $\varphi(\omega, 0) = 0$ then the random operator $\mathcal{T} : \Omega \times \mathcal{S} \to \mathcal{S}$ is called a D-Lipschitzician. Here are some particular situations $\varphi_\omega(r) = \alpha(\omega)r$, for $\alpha(\omega) > 0$ for every $\omega \in \Omega$ then the random operator $\mathcal{T}(\omega)$ is called Lipschitz with Lipschitz constant $\alpha(\omega)$. In particular, the random operator $\mathcal{T}(\omega)$ is a contraction with a contraction constant $\alpha(\omega)$, specifically for each $\omega \in \Omega$ if $\alpha(\omega) < 1$. Furthermore, $\mathcal{T}(\omega)$ is referred to as a non-linear $\mathcal{D}$-contraction random operator on $\mathcal{S}$ if $\varphi_\omega(r) = \varphi(\omega, r) < r$ for $r > 0$ and for every $\omega \in \Omega$.

Theorem 3.14. [7]*: The closed and bounded set $\mathcal{D}$ is a subset of a separable Banach algebra $\mathcal{S}$. Let $\mathcal{P}(\omega), \mathcal{Q}(\omega) : \Omega \times \mathcal{D} \to \mathcal{S}$ be two random operators that satisfy for each $\omega \in \Omega$.*

- *$\mathcal{P}(\omega)$ is D-Lipschitz with D-function φ*
- *$\mathcal{Q}(\omega)$ is completely continuous, and*
- *$\mathcal{P}(\omega)x\mathcal{Q}(\omega)x \in \mathcal{D}$ for each $x \in \mathcal{D}$*

Then the random equation

$$\mathcal{P}(\omega)x\mathcal{Q}(\omega)x = x$$

has random solutions whenever $\mathcal{M}(\omega)\varphi_\omega(r) < r,\ r > 0$, for each $\omega \in \Omega$ where $\mathcal{M}(\omega) = \|\mathcal{Q}(\omega)(\mathcal{D})\|$.

Corollary 3.15. [7]*: The closed, convex and bounded set $\mathcal{D}$ is a subset of a separable Banach algebra $\mathcal{S}$. Let $\mathcal{P}(\omega), \mathcal{Q}(\omega) : \Omega \times \mathcal{D} \to \mathcal{S}$ be two random operators which satisfy for each $\omega \in \Omega$.*

- *$\mathcal{P}(\omega)$ is Lipschitz with Lipschitz constant $\alpha(\omega)$*
- *$\mathcal{Q}(\omega)$ is continuous and compact*
- *$\mathcal{P}(\omega)x\mathcal{Q}(\omega)x \in \mathcal{D}$ for each $x \in \mathcal{D}$*

Then the random equation

$$\mathcal{P}(\omega)x\mathcal{Q}(\omega)x = x$$

has random solutions and the set of such solutions is compact whenever $\alpha(\omega)\mathcal{M}(\omega) < 1$, for each $\omega \in \Omega$ where $\mathcal{M}(\omega) = \|\mathcal{Q}(\omega)(\mathcal{D})\|$.

4. Main Results

We are searching for random solutions to FONRDE (2.1) in the space $BM(\mathcal{R}_+, \mathcal{R})$ of bounded measurable and real valued functions on $\mathcal{R}_+$. The supremum norm $\|.\|$, which is defined by

$$\|x\| = \sup_{t \in \mathcal{R}_+} | x(t) |,$$

is applied to the space $BM(\mathcal{R}_+, \mathcal{R})$. It is evident that $BM(\mathcal{R}_+, \mathcal{R})$ becomes a Banach space with regard to the previously mentioned norm for $t \in \mathcal{R}_+$. $L^1(\mathcal{R}_+, \mathcal{R})$ norm is defined by

$$\|x\|_{L^1} = \int_0^t | x(s) | \ ds$$

.

Definition 4.1. [6]: The condition of $L(\omega)$-caratheodory is satisfied by a mapping $\beta : \mathcal{R}_+ \times \mathcal{R} \times \Omega \to \mathcal{R}$ if

- $(t, \omega) \to \beta(t, x, \omega)$ is measurable for all $x \in \mathcal{R}$
- $x \to \beta(t, x, \omega)$ is continuous for all $t \in \mathcal{R}_+,\ \omega \in \Omega$
  Furthermore $L^1(\omega)$-caratheodory if
- There exists a function $q : \Omega \to L^1(\mathcal{R}_+)$ such that $| \beta(t, x, \omega) | \leq q(t, \omega)$ *a.e.*, for all $t \in \mathcal{R}_+,\ x \in \mathcal{R}$.

we need following some hypothesis for existence solution of FONRDE (2.1)

(H_0) The function $\mathcal{J} : \mathcal{R}_+ \times \mathcal{R} \times \Omega \to \mathcal{R} - \{0\}$ is measurable.

(H_1) The function $(t,\omega) \to \mathcal{K}(t,x,\omega)$ is measurable for all $x \in \mathcal{R}$.

(H_2) The function $\mathcal{J} : \mathcal{R}_+ \times \mathcal{R} \times \Omega \to \mathcal{R} - \{0\}$ is continuous function and there exist a bounded function $\alpha : \mathcal{R}_+ \times \Omega \to \mathcal{R}_+$ such that

$$\mid \mathcal{J}(t,x(t,\omega),\omega) - \mathcal{J}(t,y(t,\omega),\omega) \mid \leq \alpha(t,\omega) \mid x(t,\omega) - y(t,\omega) \mid a.e.$$

For all $t \in \mathcal{R}_+$, with $x, y \in \mathcal{R}$ and $\mid \alpha(t,\omega) \mid < 1$.

(H_3) There exists function $\phi : \mathcal{R}_+ \times \Omega \to \mathcal{R}_+$ such that for each $\omega \in \Omega$, $\mid \mathcal{J}(t,x(t,\omega),\omega) \mid \leq \phi(t,\omega)$ *a.e.*, $t \in \mathcal{R}_+$ and $\lim_{t\to\infty} \phi(t,\omega) = 0$.

(H_4) The function $\mathcal{K}$ is $L^1(\omega)$-caratheodory function. Then there exists a function $\kappa : \Omega \to L^1(\mathcal{R}_+, \mathcal{R})$ with $\kappa(t,\omega) > 0$ a.e. such that for every $\omega \in \Omega$, $\mid \mathcal{K}(t,x,\omega) \mid \leq \kappa(t,\omega)$ a.e. for $(t,x) \in \mathcal{R}_+ \times \mathcal{R}$.

Remark 4.2. The function $\xi : \mathcal{R}_+ \times \Omega \to \mathcal{R}_+$ defined by $\xi(t,\omega) = \int_0^t (t-s)^{\zeta-1}\kappa(s,\omega)\,ds$ is bounded on R_+ with bound $\Upsilon_1 = \sup\{\frac{1}{\Gamma(\zeta)}\xi(t,\omega) : t \in R_+,\ \omega \in \Omega\} > 0$

Lemma 4.3. *If $\mathcal{J}, \mathcal{K}$ satisfies FONRDE* (2.1) *then x is solution of the FONRDE* (2.1) *if and only if it is the solution of integral equation*

$$x(t,\omega) = \mathcal{J}(t,x(t,\omega),\omega)\frac{1}{\Gamma(\zeta)}\int_0^t (t-s)^{\zeta-1}\mathcal{K}(s,x(s,\omega),\omega)\,ds \tag{4.1}$$

For $t \in \mathcal{R}_+,\ \omega \in \Omega$.

Proof. FONRDE (2.1) is

$$D^{\zeta}\left(\frac{x(t,\omega)}{\mathcal{J}(t,x(t,\omega),\omega)}\right) = \mathcal{K}(t,x(t,\omega),\omega)$$
$$x(0,\omega) = 0$$

Integrating equation (2.1) from 0 to t we have

$$I^{\zeta}D^{\zeta}\left(\frac{x(t,\omega)}{\mathcal{J}(t,x(t,\omega),\omega)}\right) = I^{\zeta}\mathcal{K}(t,x(t,\omega),\omega)$$

$$\left(\frac{x(t,\omega)}{\mathcal{J}(s,x(s,\omega),\omega)}\right) = \frac{1}{\Gamma(\zeta)}\int_0^t (t-s)^{\zeta-1}\mathcal{K}(s,x(s,\omega),\omega)\,ds \tag{4.2}$$

$$x(t,\omega) = \mathcal{J}(t,x(t,\omega),\omega)\frac{1}{\Gamma(\zeta)}\int_0^t (t-s)^{\zeta-1}\mathcal{K}(s,x(s,\omega),\omega)\,ds$$

this is required equation (4.1)

Conversely, Differentiate equation (4.2) we get

$$D^{\zeta}\left(\frac{x(t,\omega)}{\mathcal{J}(t,x(t,\omega),\omega)}\right) = D^{\zeta}\frac{1}{\Gamma(\zeta)}\int_0^t (t-s)^{\zeta-1}\mathcal{K}(s,x(s,\omega),\omega)\,ds$$

$$D^{\zeta}\left(\frac{x(t,\omega)}{\mathcal{J}(t,x(t,\omega),\omega)}\right) = \mathcal{K}(t,x(t,\omega),\omega)$$

Put $t = 0$ in equation (4.1) then
$x(0,\omega) = 0$. □

Theorem 4.4. *In the event where $\|\alpha(\omega)\|\Upsilon_1 < 1$ and the hypothesis $(H_0) - (H_5)$ is true, then the FONRDE* (2.1) *has random solutions in $\mathcal{R}_+$ and Additionally, the random solutions are locally attractive on $\mathcal{R}_+$.*

Proof. Let $\mathcal{S} = BM(\mathcal{R}_+, \mathcal{R})$ be a separable Banach algebra and define a subset $\mathcal{D}$ of $\mathcal{S}$ as

$$\mathcal{D} = \{x \in \mathcal{S} : \|x\| \leq r\}$$

Where, $r = \|\phi(\omega)\|\Upsilon_1$
Clearly $\mathcal{D}$ is a closed, convex and bounded subset of $\mathcal{S}$.
Now we define the two operators $\mathcal{P}$ and $\mathcal{Q}$ on $\mathcal{D}$ i.e. $\mathcal{P}, \mathcal{Q} : \Omega \times \mathcal{D} \to \mathcal{S}$ defined by

$$\mathcal{P}(\omega)x(t) = \mathcal{J}(t, x(t, \omega), \omega)$$
$$\mathcal{Q}(\omega)x(t) = \frac{1}{\Gamma\zeta} \int_0^t (t-s)^{\zeta-1} \mathcal{K}(s, x(s, \omega), \omega)\, ds$$

For all $t \in \mathcal{R}_+$ and $\omega \in \Omega$
Then NRIE (4.1) is transformed in to the random operator equation

$$x(t, \omega) = \mathcal{P}(\omega)x(t)\mathcal{Q}(\omega)x(t)$$

For all $t \in \mathcal{R}_+$ and $\omega \in \Omega$.
We shall show that the operators $\mathcal{P}(\omega)$ and $\mathcal{Q}(\omega)$ satisfies all conditions of corollary (3.15) on $\mathcal{D}$. This will be done in following steps.
Step I: First we show that $\mathcal{P}(\omega)$ and $\mathcal{Q}(\omega)$ are random operators on $\mathcal{D}$
Since, by hypothesis (H_0) a function $\mathcal{J}(t, x(t, \omega), \omega)$ is measurable in ω for all $t \in \mathcal{R}_+$ and $x \in \mathcal{R}$ then the function $\omega \to \mathcal{P}(\omega)x$ is measurable for all $x \in \mathcal{R}$. Hence $\mathcal{P}(\omega)$ is random operator on $\mathcal{D}$.
By hypothesis H_1 the function $(t, \omega) \to \mathcal{K}(t, x, \omega)$ is measurable for all $t \in \mathcal{R}_+$ and
$x \in \mathcal{R}$. The product $(t-s)^{\zeta-1}\mathcal{K}(s, x(s, \omega), \omega)$ of continuous and measurable function is again measurable.
We know that Riemann integral as a limit of a finite sum of measurable function is again measurable.
Therefore, the function $\omega \to \frac{1}{\Gamma\zeta} \int_0^t (t-s)^{\zeta-1}\mathcal{K}(s, x(s, \omega), \omega)\, ds$ is measurable.
Hence $\mathcal{Q}(\omega)$ is random operator on $\mathcal{D}$.
Step II: Next we show that $\mathcal{P}(\omega)$ is a Lipschitz random operator on $\mathcal{D}$
Let $x, y \in \mathcal{D}$

$$\begin{aligned} | \mathcal{P}(\omega)x(t) - \mathcal{P}(\omega)y(t) | &= | \mathcal{J}(t, x(t, \omega), \omega) - \mathcal{J}(t, y(t, \omega), \omega) \\ &\leq \alpha(t, \omega) \mid x(t, \omega) - y(t, \omega) \mid \end{aligned}$$

Taking the maximum over t in the above inequality, we obtain

$$\|\mathcal{P}(\omega)x - \mathcal{P}(\omega)y\| \leq \|\alpha(\omega)\|\|x(\omega) - y(\omega)\|$$

for all $t \in \mathcal{R}_+$ and $x, y \in \mathcal{D}$ and $\omega \in \Omega$. This shows that $\mathcal{P}(\omega)$ is a Lipschitzician random operator on $\mathcal{D}$ with Lipschitz constant $\|\alpha(\omega)\|$.
Step III: Now to show that $\mathcal{Q}(\omega)$ is continuous.
Let $\{x_n\}$ be convergent sequence of points in $\mathcal{D}$ converging to point x in $\mathcal{D}$ as $n \to \infty$. Then

by Lebesgue dominated converging theorem, for all $t \in \mathcal{R}_+$ we have

$$\begin{aligned}
\lim_{n\to\infty} \mathcal{Q}(\omega)x_n(t) &= \lim_{n\to\infty} \left(\frac{1}{\Gamma\zeta} \int_0^t (t-s)^{\zeta-1} \mathcal{K}(s, x_n(s,\omega), \omega)\, ds \right) \\
&= \frac{1}{\Gamma\zeta} \int_0^t \lim_{n\to\infty} (t-s)^{\zeta-1} \mathcal{K}(s, x_n(s,\omega), \omega))\, ds \\
&= \frac{1}{\Gamma\zeta} \int_0^t (t-s)^{\zeta-1} \mathcal{K}(s, x(s,\omega), \omega)\, ds \\
&= \mathcal{Q}(\omega)x(t)
\end{aligned}$$

for every $t \in \mathcal{R}_+$ and $\omega \in \Omega$.
Hence, $\mathcal{Q}(\omega)$ is continuous on $\mathcal{D}$.
Step IV: To show that $\mathcal{Q}(\omega)$ is compact operator on $\mathcal{D}$.
It suffices to show that $\mathcal{Q}(\omega)(\mathcal{D})$ is uniformly bounded and equicontinuous set in $\mathcal{S}$, for each $\omega \in \Omega$.
First we show that $\mathcal{Q}(\omega)(\mathcal{D})$ is uniformly bounded for each $\omega \in \Omega$.
Let $x \in \mathcal{D}$ be arbitrary thus by hypothesis H_4, $\mathcal{K}$ is $L^1(\omega)$-caratheodory

$$\begin{aligned}
\mid \mathcal{Q}(\omega)x(t) \mid &= \left| \frac{1}{\Gamma\zeta} \int_0^t (t-s)^{\zeta-1} \mathcal{K}(s, x(s,\omega), \omega)\, ds \right| \\
&\leq \frac{1}{\Gamma\zeta} \int_0^t (t-s)^{\zeta-1} \mid \mathcal{K}(s, x(s,\omega), \omega) \mid \, ds \\
&\leq \frac{1}{\Gamma\zeta} \int_0^t (t-s)^{\zeta-1} \kappa(s,\omega)\, ds \\
&\leq \frac{1}{\Gamma\zeta} \xi(t,\omega)
\end{aligned}$$

Taking supremum all over t,

$$\|\mathcal{Q}(\omega)x\| \leq \Upsilon_1$$

This shows that $\mathcal{Q}(\omega)(\mathcal{D})$ is uniformly bounded subset of $\mathcal{S}$ for each $\omega \in \Omega$.
Now we show that $\mathcal{Q}(\omega)(\mathcal{D})$ is equicontinuous set in $\mathcal{S}$ for each $\omega \in \Omega$
$t, \tau \in \mathcal{R}_+$ then for any $x \in \mathcal{D}$,

$$\begin{aligned}
\mid \mathcal{Q}(\omega)x(t) - \mathcal{Q}(\omega)x(\tau) \mid &= \left| \frac{1}{\Gamma\zeta} \int_0^t (t-s)^{\zeta-1} \mathcal{K}(s, x(s,\omega), \omega)\, ds \right. \\
&\quad \left. - \frac{1}{\Gamma\zeta} \int_0^\tau (\tau-s)^{\zeta-1} \mathcal{K}(s, x(s,\omega), \omega)\, ds \right| \\
&\leq \left| \frac{1}{\Gamma\zeta} \int_0^t (t-s)^{(\zeta-1)} \kappa(s,\omega)\, ds - \frac{1}{\Gamma\zeta} \int_0^\tau (\tau-s)^{(\zeta-1)} \kappa(s,\omega)\, ds \right| \\
&\leq \mid \xi(t,\omega) - \xi(\tau,\omega) \mid
\end{aligned}$$

Since

$$\xi(t,\omega) = \frac{1}{\Gamma\zeta} \int_0^t (t-s)^{\zeta-1} \kappa(s,\omega)\, ds$$

and ξ is uniformly continuous function on $\mathcal{R}_+$.
Hence

$$\mid \mathcal{Q}(\omega)x(t) - \mathcal{Q}(\omega)x(\tau) \mid \to 0$$

as $t \to \tau$ for all $t, \tau \in \mathcal{R}_+$ and $\omega \in \Omega$.
Therefore, $\mathcal{Q}(\omega)(\mathcal{D})$ is equicontinuous set in $\mathcal{S}$. Hence by Arzela-Ascoli theorem $\mathcal{Q}(\omega)$ is

compact operator on $\mathcal{D}$.

Step V: Now

$$\begin{aligned}
\|\mathcal{P}(\omega)x\mathcal{Q}(\omega)x\| &= \|\mathcal{P}(\omega)x\|\|\mathcal{Q}(\omega)x\| \\
&= \left\{\sup_{t\in\mathcal{R}_+} \mid \mathcal{P}(\omega)x(t) \mid\right\}\left\{\sup_{t\in\mathcal{R}_+} \mid \mathcal{Q}(\omega)x(t) \mid\right\} \\
&= \left\{\sup_{t\in\mathcal{R}_+} \mid \mathcal{J}(t,x(t,\omega),\omega) \mid\right\} \\
&\quad \left\{\sup_{t\in\mathcal{R}_+}\left|\frac{1}{\Gamma\zeta}\int_0^t (t-s)^{\zeta-1}\mathcal{K}(s,x(s,\omega),\omega)\,ds\right|\right\} \\
&\leq \left\{\sup_{t\in\mathcal{R}_+} \mid \mathcal{J}(t,x(t,\omega),\omega) \mid\right\} \\
&\quad \left\{\sup_{t\in\mathcal{R}_+}\frac{1}{\Gamma\zeta}\int_0^t (t-s)^{\zeta-1} \mid \mathcal{K}(s,x(s,\omega),\omega) \mid \,ds\right\} \\
&\leq \left\{\sup_{t\in\mathcal{R}_+} \mid \mathcal{J}(t,x(t,\omega),\omega) \mid\right\} \\
&\quad \left\{\sup_{t\in\mathcal{R}_+}\frac{1}{\Gamma\zeta}\int_0^t (t-s)^{\zeta-1}\kappa(s,\omega)\,ds\right\} \\
&\leq \left\{\sup_{t\in\mathcal{R}_+} \mid \mathcal{J}(t,x(t,\omega),\omega) \mid\right\}\left\{\sup_{t\in\mathcal{R}_+}\frac{1}{\Gamma\zeta}\xi(t,\omega)\right\} \\
&\leq \|\phi(\omega)\|\Upsilon_1 \\
&\leq r
\end{aligned}$$

for each $x \in \mathcal{D}$ and $\omega \in \Omega$, so $\mathcal{P}(\omega)x\mathcal{Q}(\omega)x \in \mathcal{D}$.

Also we have

$$\begin{aligned}
\mathcal{M}(\omega) &= \|\mathcal{Q}(\omega)(\mathcal{D})\| \\
&= \sup_{x\in\mathcal{D}}\{\|\mathcal{Q}(\omega)x\| \ : x \in \mathcal{D}\} \\
&= \sup_{x\in\mathcal{D}}\left\{\sup_{t\in\mathcal{R}_+} \mid \mathcal{Q}(\omega)x(t) \mid\right\}
\end{aligned}$$

$$\begin{aligned}
&= \sup_{x\in\mathcal{D}}\left\{\sup_{t\in\mathcal{R}_+}\left|\frac{1}{\Gamma\zeta}\int_0^t (t-s)^{\zeta-1}\mathcal{K}(s,x(s,\omega),\omega)\,ds\right|\right\} \\
&\leq \sup_{x\in\mathcal{D}}\left\{\sup_{t\in\mathcal{R}_+}\frac{1}{\Gamma\zeta}\int_0^t \mid (t-s)^{\zeta-1}\mathcal{K}(s,x(s,\omega),\omega) \mid \,ds\right\} \\
&\leq \sup\left\{\sup_{t\in\mathcal{R}_+}\frac{1}{\Gamma\zeta}\int_0^t (t-s)^{\zeta-1}\kappa(s,\omega)\,ds\right\} \\
&\leq \sup\left\{\sup_{t\in\mathcal{R}_+}\frac{1}{\Gamma\zeta}\xi(t,\omega)\right\} \\
&\leq \Upsilon_1
\end{aligned}$$

And therefore $\|\alpha(\omega)\|\mathcal{M}(\omega) \leq \|\alpha(\omega)\|\Upsilon_1 < 1$, for each $\omega \in \Omega$.
Thus all the conditions of corollary (3.15) satisfied hence random operator equation $\mathcal{P}(\omega)x\mathcal{Q}(\omega)x = x$ has random solutions. This further implies that the random differential equation (2.1) has random solutions on $\mathcal{R}_+$.

Step VI: Finally, we show that the locally attractivity of the solutions.
Let x and y be any two solutions of the integral equation (4.1) in $\mathcal{D}$, then we have

$$\begin{aligned}
\mid x(t,\omega) - y(t,\omega) \mid &= \Big| \mathcal{J}(t, x(t,\omega), \omega) \frac{1}{\Gamma\zeta} \int_0^t (t-s)^{\zeta-1} \mathcal{K}(s, x(s,\omega), \omega)\, ds \\
&\quad - \mathcal{J}(t, y(t,\omega), \omega) \frac{1}{\Gamma\zeta} \int_0^t (t-s)^{\zeta-1} \mathcal{K}(s, y(s,\omega), \omega)\, ds \Big| \\
&\leq \Big| \mathcal{J}(t, x(t,\omega), \omega) \frac{1}{\Gamma\zeta} \int_0^t (t-s)^{\zeta-1} \mathcal{K}(s, x(s,\omega), \omega)\, ds \Big| \\
&\quad + \Big| \mathcal{J}(t, y(t,\omega), \omega) \frac{1}{\Gamma\zeta} \int_0^t (t-s)^{\zeta-1} \mathcal{K}(s, y(s,\omega), \omega)\, ds \Big| \\
&\leq \Big| \mathcal{J}(t, x(t,\omega), \omega) \Big| \Big| \frac{1}{\Gamma\zeta} \int_0^t (t-s)^{\zeta-1} \mathcal{K}(s, x(s,\omega), \omega)\, ds \Big| \\
&\quad + \Big| \mathcal{J}(t, y(t,\omega), \omega) \Big| \Big| \frac{1}{\Gamma\zeta} \int_0^t (t-s)^{\zeta-1} \mathcal{K}(s, y(s,\omega), \omega)\, ds \Big| \\
&\leq \Big| \mathcal{J}(t, x(t,\omega), \omega) \Big| \frac{1}{\Gamma\zeta} \int_0^t (t-s)^{\zeta-1} \kappa(s,\omega)\, ds \\
&\quad + \Big| \mathcal{J}(t, y(t,\omega), \omega) \Big| \frac{1}{\Gamma\zeta} \int_0^t (t-s)^{\zeta-1} \kappa(s,\omega)\, ds \\
&\leq 2\phi(t,\omega)\xi(t,\omega)
\end{aligned}$$

Taking the limit as $t \to \infty$ yields

$$\begin{aligned}
&\lim_{t\to\infty} \mid x(t,\omega) - y(t,\omega) \mid \leq 2 \lim_{t\to\infty} \phi(t,\omega)\xi(t,\omega) \\
&\lim_{t\to\infty} \mid x(t,\omega) - y(t,\omega) \mid = 0 \\
&\qquad \because \lim_{t\to\infty} \phi(t,\omega) = 0
\end{aligned}$$

Consequently, the FONRDE (2.1) has random solutions and all the solutions are locally attractive on $\mathcal{R}_+$. $\square$

Example 4.5. Let us consider FONRDE

$$D^{1/2}\left(\frac{x(t,\omega)}{\frac{e^{-t}cosx}{2}} \right) = \frac{e^{-t}sin\omega}{4}$$

$$x(0,\omega) = 0 \tag{4.3}$$

and $\Omega = \mathcal{R}$.

Step-by-step Hypothesis Verification

Here, $\mathcal{J}(t, x(t,\omega), \omega) = \dfrac{e^{-t}cosx}{2}$ & $\mathcal{K}(t, x(t,\omega), \omega) = \dfrac{e^{-t}sin\omega}{4}$

The functions $\mathcal{J}(t,x,\omega)$ & $\mathcal{K}(t,x,\omega)$ are continuous. Hence, both the functions are measurable. Therefore hypothesis (H_0) & (H_1) satisfied.

Now, for (H_2)

$$\begin{aligned}
\mid \mathcal{J}(t,x,\omega) - \mathcal{J}(t,y,\omega) \mid &= \left| \frac{e^{-t}cosx}{2} - \left(\frac{e^{-t}cosy}{2}\right) \right| \\
&= \frac{e^{-t}}{2} \mid cosx - cosy \mid \\
&\leq \frac{e^{-t}}{2} \mid x - y \mid
\end{aligned}$$

$\alpha(t,\omega) = \frac{e^{-t}}{2}$ with $\mid \alpha(t,\omega) \mid < 1$. Hence, hypothesis (H_2) satisfied.

$$\begin{aligned}
\mid \mathcal{J}(t,x,\omega) \mid &= \left| \frac{e^{-t}cosx}{2} \right| \\
&\leq \frac{e^{-t}}{2}
\end{aligned} \tag{4.4}$$

Therefore $\phi(t,\omega) = \dfrac{e^{-t}}{2}$ and hypothesis (H_3) satisfied.

$$\mid \mathcal{K}(t,x,\omega) \mid = \left| \frac{e^{-t}sin\omega}{4} \right| \leq \frac{e^{-t}}{4} \tag{4.5}$$

Hence, $\kappa(t,\omega) = \dfrac{e^{-t}}{4}$ is bounded measurable and integrable. Then, hypothesis (H_4) satisfied.
$\Upsilon_1 = \sup\{\dfrac{1}{\Gamma(\zeta)}\xi(t,\omega) : t \in R_+, \omega \in \Omega\} > 0$

$$\begin{aligned}
\xi(t,\omega) &= \int_0^t (t-s)^{1/2-1}\kappa(s,\omega)\,ds \\
&= \int_0^t (t-s)^{1/2-1}\frac{e^{-s}}{4}\,ds \\
&= \frac{1}{4}\int_0^t (t-s)^{1/2-1}e^{-s}\,ds
\end{aligned}$$

$$\begin{aligned}
\frac{1}{4\Gamma(1/2)}\int_0^t (t-s)^{1/2-1}e^{-s}\,ds &\leq \frac{1}{4\Gamma(1/2)}\int_0^\infty (s)^{1/2-1}e^{-s}\,ds \\
&\leq \frac{1}{4}
\end{aligned} \tag{4.6}$$

Hence, $\Upsilon_1 = \dfrac{1}{4}$
$\|\alpha(\omega)\|\Upsilon_1 = \dfrac{1}{2}\dfrac{1}{4} = \dfrac{1}{8} < 1$
Hence, from theorem (4.4) the FONRDE (4.3) has random solutions.

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