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Research Paper

ENUMERATING GROUP ACTIONS OF FINITE CYCLIC GROUPS ON FINITE SETS

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ABSTRACT

This paper provides a systematic enumeration of actions of a finite cyclic group $\mathbb{Z}/n\mathbb{Z}$ on a finite set of size k. We prove that the total number of such actions equals the number of permutations in the symmetric group S_k whose order divides n, and we give an explicit combinatorial sum over cycle type vectors. Furthermore, we show that the number of isomorphism classes of such actions equals the number $p_{D(n)}(k)$ of integer partitions of k into parts drawn from the divisor set $D(n)$ of n. The results connect elementary number theory, partition theory, and permutation group combinatorics. We also briefly discuss extensions to free groups and elementary abelian p-groups, identifying obstacles.

1. Introduction

Group actions provide a fundamental framework for studying symmetries in mathematics, connecting algebra, combinatorics, and geometry. The enumeration of actions of a finite cyclic group $\mathbb{Z}/n\mathbb{Z}$ on a set X with $|X| = k$ is a classical problem at the interface of these fields. Such actions correspond to homomorphisms $\rho\colon \mathbb{Z}/n\mathbb{Z} \to S_k$, where S_k is the symmetric group on k letters, and isomorphism classes of actions correspond to conjugacy classes of the image of a generator under these homomorphisms.

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This enumeration appears in various contexts, including the study of permutation representations, automorphism groups of graphs and designs, and applications in coding theory and number theory (see, e.g., [5, 6]). While the core results are well-known—the total number of actions equals the number of elements of S_k of order dividing n, and the number of isomorphism classes is the number of partitions of k into parts belonging to $D(n)$—an elementary derivation with full proofs and illustrative examples serves.

Some of the results have appeared in various forms in the literature (see, e.g., Cameron [4, 5] and Stanley [12]), the present paper offers a self-contained, rigorous exposition with explicit formulas, detailed examples, and a critical discussion of possible generalizations. We collect scattered facts into a single survey, clarify precise combinatorial summaries, and identify where generalizations to non-periodic groups become fundamentally more complicated.

The structure of the paper is as follows. Section 2 recalls necessary background on group actions, homomorphisms, cycle decompositions, and partitions. Section 3 states and proves the main results: Theorem 3.1 (total number of actions) and Theorem 3.2 (isomorphism classes). Subsection 3.3 provides worked examples. Section 4 discusses extensions to free groups and $\mathbb{Z}_p \times \mathbb{Z}_p$, highlighting the increased difficulty. Section 5 concludes with open problems.

2. Preliminaries

We recall the necessary background from group theory and combinatorics. Throughout, $G = \mathbb{Z}/n\mathbb{Z} = \langle g \mid g^n = e\rangle$ denotes the cyclic group of order n, and X is a finite set with $|X| = k$. We identify the symmetric group on X with S_k.

Definition 2.1. A *group action* of G on X is a map $\phi\colon G \times X \to X$ satisfying:

1. $\phi(e, x) = x$ for all $x \in X$,
2. $\phi(gh, x) = \phi(g, \phi(h, x))$ for all $g, h \in G$ and $x \in X$.

Equivalently, such an action corresponds to a homomorphism $\rho\colon G \to S_X \cong S_k$.

Definition 2.2. For $x \in X$, the *orbit* of x is $\mathrm{Orb}(x) = \{g \cdot x \mid g \in G\}$ and the *stabilizer* is $\mathrm{Stab}(x) = \{g \in G \mid g \cdot x = x\}$. The orbit-stabilizer theorem states that $|\mathrm{Orb}(x)| \cdot |\mathrm{Stab}(x)| = |G|$ when G and X are finite. Thus, for $G = \mathbb{Z}/n\mathbb{Z}$, every orbit size divides n.

Definition 2.3. Two actions $\phi, \psi\colon G \times X \to X$ are *isomorphic* if there exists a bijection $f\colon X \to X$ such that $f(\phi(g, x)) = \psi(g, f(x))$ for all $g \in G$ and $x \in X$. Equivalently, the corresponding homomorphisms $\rho, \sigma\colon G \to S_k$ are conjugate: there exists $\tau \in S_k$ with $\sigma(g) = \tau\rho(g)\tau^{-1}$ for all $g \in G$.

Definition 2.4. The *partition function* $p(m)$ counts the number of ways to write the non-negative integer m as a sum of positive integers, disregarding order (with $p(0) = 1$). Its generating function is

$$\sum_{m=0}^{\infty} p(m)x^m = \prod_{j=1}^{\infty} (1 - x^j)^{-1}.$$

More generally, for a subset $S \subseteq \mathbb{N}$, let $p_S(k)$ denote the number of partitions of k with all parts in S. In this paper, we write $p_{D(n)}(k)$ where $D(n)$ is the set of positive divisors of n.

The following lemma will be used in the extensions section.

Theorem 2.5 (Burnside's Lemma [3]). *Let a finite group H act on a finite set Y. The number of orbits is*

$$\frac{1}{|H|}\sum_{h\in H}\mathrm{Fix}(h),$$

where $\mathrm{Fix}(h) = \{y \in Y \mid h \cdot y = y\}$.

3. Main Results

3.1. **Total Number of Actions.** The total number of actions of $\mathbb{Z}/n\mathbb{Z}$ on a set of size k equals the number of homomorphisms $\mathbb{Z}/n\mathbb{Z} \to S_k$, or equivalently, the number of elements $\sigma \in S_k$ such that $\sigma^n = e$ (i.e., the order of σ divides n).

Theorem 3.1. *Let $G = \mathbb{Z}/n\mathbb{Z}$ and let X be a set with $|X| = k$. The number of group actions of G on X is*

$$\sum_{\substack{(m_d)_{d|n} \\ \sum_{d|n} d\, m_d = k}} \frac{k!}{\prod_{d|n}(d^{m_d}\, m_d!)},$$

where the sum runs over all tuples of nonnegative integers $(m_d)_{d|n}$ (one for each divisor d of n) satisfying $\sum_{d|n} d\, m_d = k$, and m_d denotes the number of cycles of length d in the cycle decomposition of a permutation in S_k.

Proof. Any action corresponds to a homomorphism $\rho\colon G \to S_k$ with $\rho(g) = \sigma \in S_k$ satisfying $\sigma^n = e$. Let σ have cycle type given by multiplicities $(m_d)_{d|n}$. The order of σ is the least common multiple of its cycle lengths. This lcm divides n if and only if every cycle length divides n. Thus, only cycle lengths in $D(n)$ are permitted, and every such permutation defines a valid homomorphism (by sending g to σ).

For a fixed tuple $(m_d)_{d|n}$ with $\sum d\, m_d = k$, the number of permutations in S_k with exactly m_d cycles of length d (for each $d \mid n$) is the standard cycle-type formula:

$$\frac{k!}{\prod_{d|n}(d^{m_d}\, m_d!)}.$$

(The factor d^{m_d} accounts for the d possible ways to write each d-cycle, and $m_d!$ accounts for the indistinguishability of cycles of the same length.) Summing over all admissible tuples yields the total number of valid σ, hence the total number of actions. ☐

3.2. **Non-Isomorphic Actions.** Two actions are isomorphic if and only if their corresponding homomorphisms are conjugate in S_k. Since G is cyclic, generated by a single element g, this reduces to the conjugacy class of $\sigma = \rho(g)$.

Theorem 3.2. *The number of group actions of $\mathbb{Z}/n\mathbb{Z}$ on a set of size k, up to isomorphism, is $p_{D(n)}(k)$, the number of partitions of k into parts belonging to $D(n)$.*

Proof. Each homomorphism is determined by the image $\sigma = \rho(g)$ with $\sigma^n = e$. Two such homomorphisms ρ_1, ρ_2 are conjugate if and only if $\rho_1(g)$ and $\rho_2(g)$ are conjugate in S_k, i.e., they have the same cycle type. The cycle type of σ is precisely a tuple $(m_d)_{d|n}$ with $\sum d\, m_d = k$. Each such tuple corresponds to a partition of k whose parts are the cycle lengths (with multiplicity m_d for each d). By the orbit-stabilizer theorem, each cycle of length d corresponds to an orbit of size d. Thus, the distinct cycle types (equivalently, the distinct partitions with parts in $D(n)$) classify the isomorphism classes. Hence there are exactly $p_{D(n)}(k)$ such classes. □

Remark 3.3. The generating function for $p_{D(n)}(k)$ is the ordinary generating function $\prod_{d|n}(1-x^d)^{-1}$.

Note 3.4. The results above are classical. The total number of actions (Theorem 3.1) is the number of elements of S_k whose order divides n, which appears in the theory of permutation representations and can be extracted from the cycle index of S_k restricted to cycle lengths in $D(n)$. Equivalently, the exponential generating function for these counts is

$$\sum_{k\geq 0} a_k \frac{x^k}{k!} = \prod_{d|n} \exp\left(\frac{x^d}{d}\right),$$

so $a_k = k!\,[x^k]\prod_{d|n}\exp(x^d/d)$ (see [5] for a detailed treatment of related generating functions). Our explicit sum formula follows directly from expanding this product via the exponential formula for permutations.

The count of isomorphism classes (Theorem 3.2) is the standard classification of cyclic permutation representations by orbit type (or cycle type of the generator). This is implicit in standard texts on permutation groups (e.g., [5, 12]) and follows immediately from the fact that conjugacy classes in S_k are determined by cycle type.

Our contribution lies in the self-contained, elementary derivation of both results, the explicit connection to the restricted partition function $p_{D(n)}(k)$, and the provision of fully worked examples with corrected computations.

3.3. **Examples.** We illustrate the theorems with concrete computations. A summary table follows the examples.

Example 3.5 ($n = 3$, $k = 5$). $D(3) = \{1, 3\}$. Partitions of 5 into $\{1, 3\}$:

- $1 + 1 + 1 + 1 + 1$: $(m_1 = 5, m_3 = 0)$
- $1 + 1 + 3$: $(m_1 = 2, m_3 = 1)$

Thus $p_{D(3)}(5) = 2$ isomorphism classes. Total actions:

$$\frac{5!}{1^5\cdot 5!} + \frac{5!}{1^2\cdot 2!\cdot 3^1\cdot 1!} = 1 + \frac{120}{2\cdot 3} = 1 + 20 = 21.$$

Example 3.6 ($n = 2$, $k = 3$). $D(2) = \{1, 2\}$. Partitions: $1 + 1 + 1$ and $1 + 2 \to p_{D(2)}(3) = 2$. Total actions:

$$\frac{3!}{1^3\cdot 3!} + \frac{3!}{1^1\cdot 1!\cdot 2^1\cdot 1!} = 1 + \frac{6}{2} = 1 + 3 = 4.$$

Example 3.7 ($n = 4$, $k = 4$). $D(4) = \{1, 2, 4\}$. Partitions:

$$1+1+1+1, \quad 1+1+2, \quad 2+2, \quad 4.$$

Hence $p_{D(4)}(4) = 4$. Total actions: compute each term

$$\begin{aligned}
(4,0,0):&\ \frac{4!}{1^4 \cdot 4!} = 1,\\
(2,1,0):&\ \frac{4!}{1^2 \cdot 2! \cdot 2^1 \cdot 1!} = \frac{24}{2 \cdot 2} = 6,\\
(0,2,0):&\ \frac{4!}{2^2 \cdot 2!} = \frac{24}{4 \cdot 2} = 3,\\
(0,0,1):&\ \frac{4!}{4^1 \cdot 1!} = \frac{24}{4} = 6.
\end{aligned}$$

Sum $1+6+3+6 = 16$. In S_4, the number of elements of order dividing 4 is: identity (1), transpositions (6), double transpositions (3), 4-cycles (6), total 16.

Table 1 summarizes the cycle-type data for the examples above.

TABLE 1. Summary of cycle types, multiplicities, and counts for the examples.

n	k	Cycle type	(m_d)	Number of permutations
3	5	1^5	$m_1 = 5$	1
3	5	$1^2 3^1$	$m_1 = 2$, $m_3 = 1$	20
2	3	1^3	$m_1 = 3$	1
2	3	$1^1 2^1$	$m_1 = 1$, $m_2 = 1$	3
4	4	1^4	$m_1 = 4$	1
4	4	$1^2 2^1$	$m_1 = 2$, $m_2 = 1$	6
4	4	2^2	$m_2 = 2$	3
4	4	4^1	$m_4 = 1$	6

4. Extensions to Other Groups

We briefly consider actions of free groups to illustrate how the presence of relations simplifies the cyclic case.

4.1. **Free groups.** Let F_n be the free group on n generators $\{s_1, \dots, s_n\}$. A homomorphism $\phi \colon F_n \to S_m$ is determined by arbitrarily assigning images $\phi(s_i) \in S_m$ to each generator (no relations to satisfy). Thus:

Proposition 4.1. *The number of group actions of F_n on a set of m elements (not up to isomorphism) is $(m!)^n$.*

Counting actions up to isomorphism requires accounting for automorphisms of the domain and conjugacy in the codomain. The group $\mathrm{Aut}(F_n) \times S_m$ acts on $\mathrm{Hom}(F_n, S_m)$ by

$$(\alpha, \sigma) \cdot \phi = \sigma^{-1} \circ (\phi \circ \alpha) \circ \sigma,$$

or equivalently, α acts by pre-composition with α^{-1} and σ by conjugation. The number of orbits (i.e., isomorphism classes of actions) is then given by Burnside's lemma 2.5:

$$\frac{1}{|\,\mathrm{Aut}(F_n)| \cdot m!} \sum_{(\alpha,\sigma)} \bigl|\mathrm{Fix}(\alpha,\sigma)\bigr|,$$

where $\mathrm{Fix}(\alpha, \sigma) = \{\phi \mid \sigma^{-1}\phi(g)\sigma = \phi(\alpha(g))$ for all $g \in F_n\}$. For $n = 1$, $F_1 \cong \mathbb{Z}$ and the formula recovers (up to the action of $\mathrm{Aut}(\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$) the cyclic case. For $n \geq 2$, $\mathrm{Aut}(F_n)$ is more complex, and explicit computation generally requires computational tools [7].

4.2. **Elementary abelian p-groups.** Consider $G = \mathbb{Z}_p \times \mathbb{Z}_p$ (p prime). A homomorphism $\phi : G \to S_m$ must send each generator to a permutation of order dividing p, and the two images must commute. Unlike the cyclic case, commuting permutations need not share a common cycle refinement in a simple partition-theoretic way. For small parameters one can enumerate, e.g., for $p = 5$, $m = 5$: permutations of order dividing 5 in S_5 are identity (1) and 5-cycles (24/5 Actually number of 5-cycles in S_5 is $5!/5 = 24$). So there are 25 such permutations. The number of commuting pairs in that set is not simply 25^2 because commuting condition imposes constraints. Thus the problem quickly becomes intricate, and no simple formula like $p_{D(p)}(m)$ holds.

5. Conclusions

We have given a rigorous enumeration of actions of a finite cyclic group on a finite set, both total and up to isomorphism. The total count is a sum over cycle types with parts in $D(n)$, and the isomorphism classes correspond to partitions with parts in $D(n)$. These results are classical but often presented without full detail; our contribution is a self-contained exposition with corrected examples and explicit generating functions. Future problems include: Find a closed form for $p_{D(n)}(k)$ in terms of divisor functions, Extend the isomorphism classification to actions of direct products $\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$. And Develop computational tools for the Burnside sum for free groups. The examples are linked to the paper "Computation of NM-polnomial and topological indices for cycle related graphs" [8]. Its application can be seen in this paper.

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