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Research Paper

# NORMALIZED DISTANCE LAPLACIAN ENERGY CHANGE DUE TO EDGE DELETION IN COMPLETE GRAPH AND COMPLETE MULTIPARTITE GRAPH

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ABSTRACT

Let $G$ be a connected graph with a distance matrix $\mathcal{D}$. The eigenvalues of $\mathcal{D}$ forms the distance spectrum or $\mathcal{D}-$ spectrum of $G$. The transmission of a vertex $v$ in $G$ is the sum of the distances from $v$ to all other vertices of $G$ and $T(G)$ is the diagonal matrix with transmission of the vertices as diagonal entries. The distance Laplacian matrix is defined as $\mathcal{D}^L(G) = T(G) - \mathcal{D}(G)$ and the normalized distance Laplacian matrix as $\mathcal{D}^{\mathcal{L}}(G) = T(G)^{-1/2} D^L(G) T(G)^{-1/2}$. If $\rho_1^{\mathcal{L}}, \rho_2^{\mathcal{L}}, \cdots, \rho_n^{\mathcal{L}}$ are $\mathcal{D}^{\mathcal{L}}$ eigenvalues of a graph $G$, then the normalized distance Laplacian energy is defined as $\mathcal{D}^{\mathcal{L}}E(G) = \sum_{i=1}^{n} |\rho_i^{\mathcal{L}} - 1|$. Let $\mathcal{D}^{\mathcal{L}}E(G-e)$ be the normalized distance Laplacian energy when an edge $e$ from a graph $G$ is removed. In this paper we are analyzing the normalized distance Laplacian energy change due to an edge deletion.

## 1. Introduction

The spectral analysis of graphs, an essential area of research in algebraic graph theory, investigates the various spectra associated with graphs. A central problem in this field is determining which graphs can be uniquely identified by their spectra.

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The normalized Laplacian matrix, introduced by Chung [1], plays a crucial role in various applications, including random walks on graphs. The eigenvalues of the normalized Laplacian matrix exhibit significant correlations with numerous graph invariants, surpassing the capabilities of eigenvalues derived from other matrices, such as the adjacency matrix. Consequently, computing the normalized Laplacian spectrum and formulating the normalized characteristic polynomial for a given graph are fundamental tasks in spectral graph theory.

In a similar vein, Reinhart [23] introduced the normalized distance Laplacian matrix, exploring its properties and extending the framework of normalized Laplacian matrices. Given the inherent difficulty in finding all the zeros of the characteristic polynomial for matrices of order greater than four, the investigation of spectral properties of matrices associated with graphs presents a challenging yet fascinating endeavor.

All the graphs $G$ considering are simple and connected. Denote by $D(G)$ the diagonal matrix of the vertex degrees of $G$ called the degree matrix of $G$. The adjacency matrix $A(G) = [a_{ij}]$, $1 \leq i, j \leq n$ is defined by $a_{ij} = 1$ if $v_i$ adjacent to $v_j$ and $a_{ij} = 0$ otherwise. The normalized Laplacian matrix of a graph $G$ is introduced in [1] as the square matrix with rows and columns indexed by $1 \leq i, j \leq n$ of $G$, denoted as $\mathcal{L}(G)$. For any two vertices $v_i$ and $v_j$ of $G$, the $(i, j)^{th}$ entry of $\mathcal{L}(G)$

$$\mathcal{L}_{ij}(G) = \begin{cases} 1 & \text{if } v_i = v_j \text{ and } d_{v_i} \neq 0 \\ \frac{-1}{\sqrt{d_{v_i} d_{v_j}}} & \text{if } v_i \text{ and } v_j \text{ are adjacent} \\ 0 & otherwise \end{cases}$$

where $d_{v_i}$ and $d_{v_j}$ are the degrees of the vertices $v_i$ and $v_j$ respectively.

We can also write $\mathcal{L}(G) = D^{-1/2}(G)L(G)D^{-1/2}(G) = I - D^{-1/2}(G)A(G)D^{-1/2}(G)$ where $A(G)$, $L(G)$ and $D(G)$ are the adjacency matrix, Laplacian matrix and degree matrix respectively of the graph $G$ with the convention that $D(G)^{-1}(u, u) = 0$ if $d_u = 0$. The eigenvalues of $\mathcal{L}(G)$ are called normalized Laplacian eigenvalues of $G$ and forms the $\mathcal{L}$- spectrum of $G$. Two graphs are said to be $\mathcal{L}$- co-spectral if they have the same $\mathcal{L}$- spectrum. In [1] Chung showed that $\mathcal{L}(G)$ is a positive definite symmetric matrix and all its eigenvalues lie within the interval $[0, 2]$ and 0 is always a normalized Laplacian eigenvalue of any graph $G$. Some recent results on normalized Laplacian eigenvalues can be seen in [15],[16], [17], [18], [19], [8].

The distance matrix $\mathcal{D}(G)$ of a graph $G$ is defined so that its $(i, j)^{th}$ entry is equal to $d(v_i, v_j)$, the distance between the vertices $v_i$ and $v_j$ of $G$. The eigenvalues of $\mathcal{D}(G)$ are said to be the $\mathcal{D}$-eigenvalues of $G$. Transmission of a vertex $v$ of $G$, $Tr(v)$ is the sum of the distances from $v$ to all other vertices of $G$ and the transmission matrix $T(G)$ is the diagonal matrix defined as $T(G) = diag(Tr(v_1), Tr(v_2)...., Tr(v_n))$. The distance Laplacian matrix is defined as $\mathcal{D}^L(G) = T(G) - \mathcal{D}(G)$. The normalized distance Laplacian matrix is defined as in [23],

$$\mathcal{D}^{\mathcal{L}}_{ij}(G) = \begin{cases} 1 & \text{if } i = j \\ \frac{-d(v_i, v_j)}{\sqrt{Tr(v_i)Tr(v_j)}} & \text{if } i \neq j \end{cases}$$

We can see that $\mathcal{D}^{\mathcal{L}}(G) = T(G)^{-1/2}\mathcal{D}^L(G)T(G)^{-1/2}$. The eigenvalues of $\mathcal{D}^{\mathcal{L}}(G)$ are called normalized distance Laplacian eigenvalues of $G$ and form the $\mathcal{D}^{\mathcal{L}}$-spectrum of $G$. Two graphs are said to be $\mathcal{D}^{\mathcal{L}}$- cospectral if they have the same $\mathcal{D}^{\mathcal{L}}$- spectrum. A graph $G$ is said to be a $r$-transmission regular if $Tr(v) = r$ for all vertices $v$ of $G$. If $G$ is $r$-transmission regular, then $\mathcal{D}^{\mathcal{L}}(G) = rI - \mathcal{D}(G)$, $\mathcal{D}^{\mathcal{L}}(G) = T(G)^{-1/2}(T(G) - \mathcal{D}(G))T(G)^{-1/2} = I - \frac{\mathcal{D}(G)}{r}$. Some recent results on normalized distance Laplacian eigenvalues are given in [13], [14], [12], [8]. If $\rho_1^{\mathcal{L}}, \rho_2^{\mathcal{L}}, \cdots, \rho_n^{\mathcal{L}}$ are $\mathcal{D}^{\mathcal{L}}$ eigenvalues of a graph $G$, then the normalized distance Laplacian energy is defined as

$$\mathcal{D}^{\mathcal{L}}E(G) = \sum_{i=1}^{n} |\rho_i^{\mathcal{L}} - 1|$$

The singular values of a rectangular matrix $M$ with complex entries, are defined to be the square roots of the eigenvalues of the positive semi-definite matrix $M^*M$, where $M^*$ is the conjugate transpose of $M$. From here on, we denote singular values of $M$ by $\sigma_i(M)$ for $i = 1, 2, ..., n$.

**Observation:** The singular values of a real symmetric matrix $M$ are the absolute values of the eigenvalues of $M$.

Thus,

$$\mathcal{D}^{\mathcal{L}}E(G) = \sum_{i=1}^{n} \sigma_i(\mathcal{D}^{\mathcal{L}} - I)$$

**Lemma 1.1.** [10] *Let $A$ and $B$ be square matrices of order $n$. Then,*

$$\sum_{i=1}^{n} \sigma_i(A + B) \leq \sum_{i=1}^{n} \sigma_i(A) + \sum_{i=1}^{n} \sigma_i(B)$$

**Lemma 1.2.** *Let $G$ and $H$ be graphs of order $n$ and $M = \mathcal{D}^{\mathcal{L}}(H) - \mathcal{D}^{\mathcal{L}}(G)$, then*

$$|\mathcal{D}^{\mathcal{L}}E(H) - \mathcal{D}^{\mathcal{L}}E(G)| \leq \sum_{i=1}^{n} \sigma_i(M)$$

*Proof.* Since $I - \mathcal{D}^{\mathcal{L}}(G) = M + I - \mathcal{D}^{\mathcal{L}}(H)$, by Lemma 1.1,

$$\sum_{i=1}^{n} \sigma_i(I - \mathcal{D}^{\mathcal{L}}(G)) \leq \sum_{i=1}^{n} \sigma_i(M) + \sum_{i=1}^{n} \sigma_i(I - \mathcal{D}^{\mathcal{L}}(H))$$

Hence $\mathcal{D}^{\mathcal{L}}E(G) - \mathcal{D}^{\mathcal{L}}E(H) \leq \sum_{i=1}^{n} \sigma_i(M)$.

Similarly applying Lemma 1.1 to $I - \mathcal{D}^{\mathcal{L}}(H) = -M + I - \mathcal{D}^{\mathcal{L}}(G)$, we have

$$\sum_{i=1}^{n} \sigma_i(I - \mathcal{D}^{\mathcal{L}}(H)) \leq \sum_{i=1}^{n} \sigma_i(-M) + \sum_{i=1}^{n} \sigma_i(I - \mathcal{D}^{\mathcal{L}}(G))$$

Hence

$$\mathcal{D}^{\mathcal{L}}E(H) - \mathcal{D}^{\mathcal{L}}E(G) \leq \sum_{i=1}^{n} \sigma_i(-M) \leq \sum_{i=1}^{n} \sigma_i(M)$$

Thus completing the proof. □

In this paper, we are analyze how the normalized distance Laplacian energy of complete graphs and complete multipartite graphs changes when an edge is deleted. A similar problem for the distance energy was studied in [5], [4]. Deleting an edge may increase or decrease the normalized distance Laplacian energy $\mathcal{D}^{\mathcal{L}}E$. We also obtain some conditions for the normalized distance Laplacian energy increases when an edge is deleted.

## 2. $\mathcal{D}^{\mathcal{L}}E$ change in $K_n$ due to an edge deletion

**Theorem 2.1.** [13] *Let $G$ be a graph of order $n \geq 2$. Then $\mathcal{D}^{\mathcal{L}}E(G) \geq 2$ with equality if and only if $G$ has one positive $\mathcal{D}$ eigenvalue.*

**Theorem 2.2.** *For $n \geq 2$, $\mathcal{D}^{\mathcal{L}}E(K_n - e) \geq \mathcal{D}^{\mathcal{L}}E(K_n)$ and $|\mathcal{D}^{\mathcal{L}}E(K_n - e) - \mathcal{D}^{\mathcal{L}}E(K_n)| \leq 1.0117$ for any edge $e$ of $K_n$.*

*Proof.* By [22], complete graphs $K_n$ have a unique positive $\mathcal{D}$ eigenvalue. Thus theorem 2.1 implies $\mathcal{D}^{\mathcal{L}}(K_n - e) \geq \mathcal{D}^{\mathcal{L}}(K_n)$.
Consider the graph $G = K_n$ and $H = K_n - e$ for $e = u_1u_2$.

$$\mathcal{D}^{\mathcal{L}}(K_n) = \begin{bmatrix} 1 & \frac{-1}{n-1} & \frac{-1}{n-1}J_{1\times(n-2)} \\ \frac{-1}{n-1} & 1 & \frac{-1}{n-1}J_{1\times(n-2)} \\ \frac{-1}{n-1}J_{(n-2)\times 1} & \frac{-1}{n-1}J_{(n-2)\times 1} & \frac{-1}{n-1}(I-J)+I \end{bmatrix}$$

When an edge $e = u_1u_2$ is deleted, $Tr_{K_n-e}(u_1) = Tr_{K_n-e}(u_2) = n$ and $Tr_{K_n-e}(u_i) = n-1$ for all other vertices.
Hence,

$$\mathcal{D}^{\mathcal{L}}(K_n - e) = \begin{bmatrix} 1 & \frac{-2}{n} & \frac{-1}{\sqrt{n(n-1)}}J_{1\times(n-2)} \\ \frac{-2}{n} & 1 & \frac{-1}{\sqrt{n(n-1)}}J_{1\times(n-2)} \\ \frac{-1}{\sqrt{n(n-1)}}J_{(n-2)\times 1} & \frac{-1}{\sqrt{n(n-1)}}J_{(n-2)\times 1} & \frac{-1}{n-1}(I-J)+I \end{bmatrix}$$

Therefore

$$M = \mathcal{D}^{\mathcal{L}}(K_n - e) - \mathcal{D}^{\mathcal{L}}(K_n) = \begin{bmatrix} 0 & x & X^t \\ x & 0 & X^t \\ X & X & 0_{(n-2)\times(n-2)} \end{bmatrix}$$

where $x = \left(\frac{-2}{n} + \frac{1}{n-1}\right)$ and $X^t = \left(\frac{1}{n-1} - \frac{1}{\sqrt{n(n-1)}}\right) J_{1\times(n-2)}$.

Since $X$ has nonzero entry, $rank(M) = 3$ ;hence 0 is an eigenvalue of $M$ with multiplicity $n-3$.

Let $\begin{bmatrix} a \\ b \\ \mathbf{v} \end{bmatrix}$ be an eigenvector of $M$ corresponding to an eigenvalue $\lambda \neq 0$.

$M \begin{bmatrix} a \\ b \\ \mathbf{v} \end{bmatrix} = \lambda \begin{bmatrix} a \\ b \\ \mathbf{v} \end{bmatrix}$ implies

$$bx + X^t\mathbf{v} = \lambda a \tag{2.1}$$

$$ax + X^t\mathbf{v} = \lambda b \tag{2.2}$$

$$aX + bX = \lambda \mathbf{v} \tag{2.3}$$

Substituting $\mathbf{v} = \frac{a+b}{\lambda}X$ in (2.1) and (2.2), we have

$$a\left(\frac{||X||^2}{\lambda} - \lambda\right) + b\left(x + \frac{||X||^2}{\lambda}\right) = 0 \tag{2.4}$$

$$a\left(x + \frac{||X||^2}{\lambda}\right) + b\left(\frac{||X||^2}{\lambda} - \lambda\right) = 0 \tag{2.5}$$

and solving these equations, we have

$$a[(x+\frac{||X||^2}{\lambda})^2-(\lambda-\frac{||X||^2}{\lambda})^2]=0$$

But, $a=0$ implies $\begin{bmatrix} a \\ b \\ \mathbf{v} \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.
Thus, we have the equation

$$\lambda^3-\lambda(x^2+2||X||^2)-2x||X||^2=0$$

which can be factored into

$$(\lambda+x)(\lambda^2-\lambda x-2||X||^2)=0$$

whose solutions are $\lambda_1=-x$, $\lambda_2=\frac{x+\sqrt{x^2+8||X||^2}}{2}$ and $\lambda_3=\frac{x-\sqrt{x^2+8||X||^2}}{2}$ which are the simple eigenvalues of $M$.
Since $\sqrt{x^2+8||X||^2}>|x|$, $\lambda_2\geq 0$ and $\lambda_3\leq 0$.

$$|\lambda_2|+|\lambda_3|=\frac{x+\sqrt{x^2+8||X||^2}}{2}-\frac{x-\sqrt{x^2+8||X||^2}}{2}=\sqrt{x^2+8||X||^2}$$

Therefore, $|\lambda_1(M)|+|\lambda_2(M)|+|\lambda_3(M)|=|x|+\sqrt{x^2+8||X||^2}$.
Thus,

$$|\mathcal{D}^{\mathcal{L}}E(K_n-e)-\mathcal{D}^{\mathcal{L}}E(K_n)|\leq\sum_{i=1}^{n}\sigma_i(M)=\lambda_1(M)+\lambda_2(M)+\lambda_3(M)=|x|+\sqrt{x^2+8||X||^2}$$

for $x=\left(\frac{-2}{n}+\frac{1}{n-1}\right)$ and $X^t=\left(\frac{1}{n-1}-\frac{1}{\sqrt{n(n-1)}}\right)J_{1\times(n-2)}$.

For $n\geq 2$, $\frac{-2}{n}+\frac{1}{n-1}=\frac{2-n}{n(n-1)}$ tends to 0 as $n\longrightarrow\infty$ and for $n\geq 3$ the numerator $2-n$ grows linearly, but the denominator $n^2-n$ grows quadratically, so maximum value of $|x|$ is at $n=3$.

Therefore $|x|\leq\frac{1}{6}$ and $x^2\leq\frac{1}{36}$

Also, $\frac{n-2}{n-1}\leq 1$ and $\left(\frac{1}{\sqrt{n-1}}-\frac{1}{\sqrt{n}}\right)^2\leq(1-\frac{1}{\sqrt{2}})^2\approx 0.0858$ implies $||X||^2=\frac{n-2}{n-1}(\frac{1}{\sqrt{n-1}}-\frac{1}{\sqrt{n}})^2\leq 0.0858$.
Hence

$$|\mathcal{D}^{\mathcal{L}}E(K_n-e)-\mathcal{D}^{\mathcal{L}}E(K_n)|\leq\frac{1}{6}+\sqrt{\frac{1}{36}+.68624}\approx 1.0117$$

□

## 3. $\mathcal{D}^{\mathcal{L}}E$ change in $K_{p,p,\ldots,p}$ due to an edge deletion

In this section, we can analyze how much the normalized distance Laplacian energy varies when an edge is removed from the complete multipartite graph $K_{p,p,\cdots,p}$.
The following result shows the connection between normalized distance Laplacian energy and Distance energy for any graph $G$.

**Theorem 3.1.** [12] *Let $G$ be a graph of order $n\geq 2$ and $Tr_{min}$ and $Tr_{max}$ be the minimum and the maximum transmission of $G$, respectively. Then,* $\frac{\mathcal{D}E(G)}{Tr_{max}}\leq\mathcal{D}^{\mathcal{L}}E(G)\leq\frac{\mathcal{D}E(G)}{Tr_{min}}$

**Theorem 3.2.** [5] *Let $G = K_{p,p,\ldots,p}$ be the complete $r$-partite graph with $r \geq 3$, $p \geq 2$, and $e$ be any edge. Then the distance eigenvalues of $H = K_{p,p,\ldots,p} - e$ are:*

- $-2$ *with multiplicity* $(p-1)r-2$,
- $p-2$ *with multiplicity* $r-3$,
- *and the roots $x_i$ $(i = 1,2,3,4,5)$ of the following polynomial:*

$$f_1(x) = g_1(x)h_1(x)$$

*where* $g_1(x) = x^2 + (-p+5)x - 3p + 7$ *and* $h_1(x) = x^3 + (-pr - 2p + 5)x^2 + (p^2 r - 3pr + p^2 - 6p + 5)x + p^2 r - pr + p^2 - 3p - 2$

**Lemma 3.3.** [5] *Let $x_3, x_4, x_5$ be the roots of $h_1(x)$ with $p \geq 2$, $r \geq 3$. Then $h_1(x)$ has exactly one negative root and two positive roots, that is:*

$$x_3 < 0 < x_4 \leq x_5.$$

**Lemma 3.4.** [5] *If $p = 2$, $r \geq 3$, then:*

$$x_3 < -1.$$

**Theorem 3.5.** *For $r \geq 3$, $\mathcal{D}^{\mathcal{L}}E(K_{2,2,\ldots,2} - e) > \mathcal{D}^{\mathcal{L}}E(K_{2,2,\ldots,2})$ and $|\mathcal{D}^{\mathcal{L}}E(K_{p,p,\ldots,p}) - \mathcal{D}^{\mathcal{L}}E(K_{p,p,\ldots,p} - e)| \leq 0.4235$ for any edge $e$ of $K_{p,p,\ldots,p}$*

*Proof.* We have,

$$\mathcal{D}E(K_{2,2,\ldots,2}) = 4r \text{ and } \mathcal{D}^{\mathcal{L}}E(K_{2,2,\ldots,2}) = 2$$

$K_{2,2,\ldots,2} - e$ have more than one positive $\mathcal{D}$ eigenvalue.
Hence,

$$\mathcal{D}^{\mathcal{L}}E(K_{2,2,\ldots,2} - e) > 2 \text{ and } \mathcal{D}^{\mathcal{L}}E(K_{2,2,\ldots,2} - e) > \mathcal{D}^{\mathcal{L}}E(K_{2,2,\ldots,2})$$

Let $\{u_1, u_2, \cdots, u_p\}, \{v_1, v_2, \cdots, v_p\}, \cdots, \{w_1, w_2, \ldots, w_p\}$ be the $r$ partite sets of $K_{p,p,\cdots,p}$ and $e = u_p v_1$. Let us partition the vertex set of $K_{p,p,\ldots,p}$ into 5 sets as follows. $V_1 = \{u_1, u_2, \cdots, u_{p-1}\}$, $V_2 = \{u_p\}$, $V_3 = \{v_1\}$, $V_4 = \{v_2, \cdots, v_p\}$ and $V_5$ be all other vertices of $K_{p,p,\cdots,p}$. For $K_{p,p,\cdots,p} - u_p v_1$, $Tr(v) = p(r-1) + 2(p-1)$ for all $v \neq u_p, v_1$ and $Tr(v) = p(r-2) + 3(p-1) + 2$ for $v = u_p$ or $v_1$.
Let $\alpha = p(r-1) + 2(p-1)$, $\beta = p(r-2) + 3(p-1) + 2 = \alpha + 1$ and hence the normalized distance Laplacian matrix of $K_{p,p,\ldots,p}$ and $K_{p,p,\ldots,p} - e$ is given by

$$\mathcal{D}^{\mathcal{L}}(K_{p,p,\ldots,p}) = \begin{bmatrix} \frac{-2(J-I)}{\alpha} + I & \frac{-2}{\alpha} J_{p-1\times 1} & \frac{-1}{\alpha} J_{p-1\times 1} & \frac{-1}{\alpha} J_{p-1\times p-1} & \frac{-1}{\alpha} J_{p-1\times (r-2)p} \\ \frac{-2}{\alpha} J_{1\times p-1} & 1 & \frac{-1}{\alpha} & \frac{-1}{\alpha} J & \frac{-1}{\alpha} J \\ \frac{-1}{\alpha} J_{1\times p-1} & \frac{-1}{\alpha} & 1 & \frac{-2}{\alpha} & \frac{-1}{\alpha} J \\ \frac{-1}{\alpha} J_{p-1\times p-1} & \frac{-1}{\alpha} J & \frac{-2}{\alpha} J & \frac{2(J-I)}{\alpha} + I & \frac{-1}{\alpha} J \\ \frac{-1}{\alpha} J_{(r-2)p\times p-1} & \frac{-1}{\alpha} J & \frac{-1}{\alpha} J & \frac{-1}{\alpha} J & \frac{2(J-I)}{\alpha} + I \end{bmatrix}$$

$\mathcal{D}^{\mathcal{L}}(K_{p,p,\ldots,p}-e)=$

$$\begin{bmatrix} \frac{-2(J-I)}{\alpha}+I & \frac{-2}{\sqrt{\alpha(\alpha+1)}}J_{p-1\times 1} & \frac{-1}{\sqrt{\alpha(\alpha+1)}}J_{p-1\times 1} & \frac{-1}{\alpha}J_{p-1\times p-1} & \frac{-1}{\alpha}J_{p-1\times(r-2)p} \\ \frac{-2}{\sqrt{\alpha(\alpha+1)}}J_{1\times p-1} & 1 & \frac{-2}{\alpha+1} & \frac{-1}{\sqrt{\alpha(\alpha+1)}}J & \frac{-1}{\sqrt{\alpha(\alpha+1)}}J \\ \frac{-1}{\sqrt{\alpha(\alpha+1)}}J_{1\times p-1} & \frac{-2}{\alpha+1} & 1 & \frac{-2}{\sqrt{\alpha(\alpha+1)}} & \frac{-1}{\sqrt{\alpha(\alpha+1)}}J \\ \frac{-1}{\alpha}J_{p-1\times p-1} & \frac{-1}{\sqrt{\alpha(\alpha+1)}}J & \frac{-2}{\sqrt{\alpha(\alpha+1)}}J & \frac{-2(J-I)}{\alpha}+I & \frac{-1}{\alpha}J \\ \frac{-1}{\alpha}J_{(r-2)p\times p-1} & \frac{-1}{\sqrt{\alpha(\alpha+1)}}J & \frac{-1}{\sqrt{\alpha(\alpha+1)}}J & \frac{-1}{\alpha}J & \frac{-2(J-I)}{\alpha}+I \end{bmatrix}$$

$$N=\mathcal{D}^{\mathcal{L}}(K_{p,p,\ldots,p}-e)-\mathcal{D}^{\mathcal{L}}(K_{p,p,\ldots,p})$$

$$=\begin{bmatrix} 0_{p-1\times p-1} & 2Y & Y & 0_{p-1\times p-1} & 0_{p-1\times(r-2)p} \\ 2Y^t & 0 & y & Y^t & Z^t \\ Y^t & y & 0 & 2Y^t & Z^t \\ 0_{p-1\times p-1} & Y & 2Y & 0_{p-1\times p-1} & 0_{p-1\times(r-2)p} \\ 0_{(r-2)p\times p-1} & Z & Z & 0_{(r-2)p\times p-1} & 0_{(r-2)p\times(r-2)p} \end{bmatrix}$$

where $y=\left(\frac{-2}{\alpha+1}+\frac{1}{\alpha}\right)$, $Y=\left(\frac{-1}{\sqrt{\alpha(\alpha+1)}}+\frac{1}{\alpha}\right)J_{p-1\times 1}$ and $Z=(\frac{-1}{\sqrt{\alpha(\alpha+1)}}+\frac{1}{\alpha})J_{(r-2)p\times 1}$. Since $Y$ has nonzero entry, $rank(N)=4$; hence 0 is an eigenvalue of $N$ with multiplicity $rp-4$.

For $\mu\neq 0$, $M\begin{pmatrix}\mathbf{u}\\ a\\ b\\ \mathbf{v}\\ \mathbf{w}\end{pmatrix}=\mu\begin{pmatrix}\mathbf{u}\\ a\\ b\\ \mathbf{v}\\ \mathbf{w}\end{pmatrix}$ where $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are respectively $p-1\times 1$, $p-1\times 1$ and $(r-2)p\times 1$ vectors gives the following equations,

$$2aY+bY=\mu\mathbf{u} \tag{3.1}$$

$$2Y^t\mathbf{u}+by+Y^t\mathbf{v}+Z^t\mathbf{w}=\mu a \tag{3.2}$$

$$Y^t\mathbf{u}+ay+2Y^t\mathbf{v}+Z^t\mathbf{w}=\mu b \tag{3.3}$$

$$aY+2bY=\mu\mathbf{v} \tag{3.4}$$

$$aZ+bZ=\mu\mathbf{w} \tag{3.5}$$

Substituting $\mathbf{u}=\frac{2a+b}{\mu}Y$, $\mathbf{v}=\frac{(a+2b)}{\mu}Y$ and $\mathbf{w}=\frac{a+b}{\mu}Z$ in equations (3.2) and (3.3) gives a four degree equation in $\mu$ whose solutions are eigenvalues of $N$.

$$\mu^4-\mu^2[y^2+2(5||Y||^2+||Z||^2)]-\mu[2y(4||Y||^2+||Z||^2)]+||Y||^2(9||Y||^2+2||Z||^2)=0$$

This fourth degree equation can be factorised into $[\mu^2+y\mu-||Y||^2][\mu^2-y\mu-(9||Y||^2+2||Z||^2)]=0$.

The solutions of the equation $[\mu^2+y\mu-||Y||^2]=0$ are

$$\mu=\frac{-y\pm\sqrt{y^2+4||Y||^2}}{2}$$

Similarly, the solutions of the equation $[\mu^2-y\mu-(9||Y||^2+2||Z||^2)]=0$ are

$$\mu=\frac{y\pm\sqrt{y^2+4(9||Y||^2+2||Z||^2)}}{2}$$

$$|y| < \sqrt{y^2+4||Y||^2} \text{ implies the roots } \mu_1 > 0 \text{ and} \mu_2 < 0$$

Hence, $|\mu_1| + |\mu_2| = \mu_1 - \mu_2 = \sqrt{y^2+4||Y||^2}$

Similarly, for the solutions $\mu_3$ and $\mu_4$ of $[\mu^2 - y\lambda - (9||Y||^2 + 2||Z||^2)] = 0$, $|\mu_3| + |\mu_4| = \sqrt{y^2 + 4(9||Y||^2 + 2||Z||^2)}$.

Hence,

$$|\mu_1| + |\mu_2| + |\mu_3| + |\mu_4| = \sqrt{y^2+4||Y||^2} + \sqrt{y^2 + 4(9|Y||^2 + 2||Z||^2)}$$

Now, we have, $\alpha = p(r-1) + 2(p-1)$, $\beta = \alpha + 1$.

$p \geq 2$ and $r \geq 2$ implies $\alpha \geq 4$ and $\beta \geq 5$.

$$y = \frac{-2}{(\alpha+1)} + \frac{1}{\alpha} \text{ implies } y^2 = (\frac{-2}{(\alpha+1)} + \frac{1}{\alpha})^2 \leq (\frac{3}{20})^2 \approx 0.0225$$

The expression $\frac{p-1}{\alpha}$ decreases as $p$ and $r$ increase, and the highest value occurs when both $p$ and $r$ are at their minimum allowed values. Thus, $\frac{p-1}{\alpha} \leq 1/4$. Also, $(\frac{1}{\sqrt{\alpha}} - \frac{1}{\sqrt{\alpha+1}})^2$ decreases as $\alpha$ increases and thus $(\frac{1}{\sqrt{\alpha}} - \frac{1}{\sqrt{\alpha+1}})^2 = (\frac{\sqrt{\alpha+1}-\sqrt{\alpha}}{\sqrt{\alpha+1}\sqrt{\alpha}})^2 = \frac{1}{[\sqrt{\alpha+1}+\sqrt{\alpha}]^2\alpha(\alpha+1)}$ has maximum value at $\alpha = 4$.

$$(\frac{1}{\sqrt{\alpha}} - \frac{1}{\sqrt{\alpha+1}})^2 \leq \frac{1}{[\sqrt{5}+\sqrt{4}]^2 20} \approx 0.00279$$

$$||Y||^2 = (p-1)(\frac{-1}{\sqrt{\alpha\beta}} + \frac{1}{\alpha})^2 = \frac{p-1}{\alpha}(\frac{\sqrt{\alpha+1}-\sqrt{\alpha}}{\sqrt{(\alpha+1)\alpha}})^2 \leq \frac{1}{4}0.00279$$

The expression $\frac{p(r-2)}{\alpha}$ approaches 1 for large $r$, but for smaller values of $p$ and $r$, the expression remains below 1.

$$||Z||^2 = p(r-2)(\frac{-1}{\sqrt{\alpha\beta}} + \frac{1}{\alpha})^2 = \frac{p(r-2)}{\alpha}(\frac{\sqrt{\alpha+1}-\sqrt{\alpha}}{\sqrt{(\alpha+1)\alpha}})^2 \leq 0.00279$$

Hence,

$$|\mathcal{D}^{\mathcal{L}}E(K_{p,p,\ldots,p}) - \mathcal{D}^{\mathcal{L}}E(K_{p,p,\ldots,p} - e)| \leq \sqrt{0.0225 + 0.00279} + \sqrt{0.0225 + 4[\frac{9}{4}0.00279 + 2(0.00279)]} \approx 0.4235$$

□

## 4. $\mathcal{D}^{\mathcal{L}}E$ change in $K_{p,p,p+1}$ due to an edge deletion

For the graph $K_{p,p,p+1}$, there are two possible scenarios for edge deletion. We begin by examining the case where the endpoints of the deleted edge belong to two vertex subsets of equal cardinality.

**Theorem 4.1.** $|\mathcal{D}^{\mathcal{L}}E(K_{p,p,p+1}) - \mathcal{D}^{\mathcal{L}}E(K_{p,p,p+1} - u_pv_1)| \leq 0.241056$ *where the endpoints of the deleted edge $e$ in $K_{p,p,p+1}$ belong to the partite sets of equal cardinality.*

*Proof.* Let $\{u_1, u_2, ..., u_p\}$, $\{v_1, v_2, ..., v_p\}$ and $\{w_1, w_2, ..., w_{p+1}\}$ be the partite sets of $K_{p,p,p+1}$ and without loss of generality let $e = u_pv_1$. By partitioning the vertex set of $K_{p,p,p+1}$ into 5 sets, $U_1 = \{u_1, u_2, ..., u_{p-1}\}$, $U_2 = \{u_p\}$, $U_3 = \{v_1\}$, $U_4 = \{v_2, ..., v_p\}$ and $U_5 = \{w_1, w_2, ..., w_{p+1}\}$. For $K_{p,p,p+1} - u_pv_1$, $Tr(v) = 4p - 1$ for all $v \in U_1, U_4$ and $Tr(v) = 4p$ for $v \in U_2, U_3, U_5$.

Thus the normalized distance Laplacian matrix of $K_{p,p,p+1}$ and $K_{p,p,p+1} - u_pv_1$ is given by

$\mathcal{D}^{\mathcal{L}}(K_{p,p,p+1}) =$

$$\begin{bmatrix} \frac{-2(J-I)}{4p-1}+I & \frac{-2}{4p-1}J_{p-1\times 1} & \frac{-1}{4p-1}J_{p-1\times 1} & \frac{-1}{4p-1}J_{p-1\times p-1} & \frac{-1}{\sqrt{4p(4p-1)}}J_{p-1\times(p+1)} \\ \frac{-2}{4p-1}J & 1 & \frac{-1}{4p-1} & \frac{-1}{4p-1}J & \frac{-1}{\sqrt{4p(4p-1)}}J \\ \frac{-1}{4p-1}J & \frac{-1}{4p-1} & 1 & \frac{-2}{4p-1} & \frac{-1}{\sqrt{4p(4p-1)}}J \\ \frac{-1}{4p-1}J & \frac{-1}{4p-1}J & \frac{-2}{4p-1}J & \frac{-2(J-I)}{4p-1}+I & \frac{-1}{\sqrt{4p(4p-1)}}J \\ \frac{-1}{\sqrt{4p(4p-1)}}J & \frac{-1}{\sqrt{4p(4p-1)}}J & \frac{-1}{\sqrt{4p(4p-1)}}J & \frac{-1}{\sqrt{4p(4p-1)}}J & \frac{-2(J-I)}{4p}+I \end{bmatrix}$$

$\mathcal{D}^{\mathcal{L}}(K_{p,p,p+1} - u_p v_1) =$

$$\begin{bmatrix} \frac{-2(J-I)}{4p-1}+I & \frac{-2}{\sqrt{4p(4p-1)}}J_{p-1\times 1} & \frac{-1}{\sqrt{4p(4p-1)}}J_{p-1\times 1} & \frac{-1}{4p-1}J_{p-1\times p-1} & \frac{-1}{\sqrt{4p(4p-1)}}J_{p-1\times(p+1)} \\ \frac{-2}{\sqrt{4p(4p-1)}}J & 1 & \frac{-2}{4p} & \frac{-1}{\sqrt{4p(4p-1)}}J & \frac{-1}{4p}J \\ \frac{-1}{\sqrt{4p(4p-1)}}J & \frac{-2}{4p} & 1 & \frac{-2}{\sqrt{4p(4p-1)}} & \frac{-1}{4p}J \\ \frac{-1}{4p-1}J & \frac{-1}{\sqrt{4p(4p-1)}}J & \frac{-2}{\sqrt{4p(4p-1)}}J & \frac{-2(J-I)}{4p-1}+I & \frac{-1}{\sqrt{4p(4p-1)}}J \\ \frac{-1}{\sqrt{4p(4p-1)}}J & \frac{-1}{4p}J & \frac{-1}{4p}J & \frac{-1}{\sqrt{4p(4p-1)}}J & \frac{-2(J-I)}{4p}+I \end{bmatrix}$$

Hence $M' = \mathcal{D}^{\mathcal{L}}(K_{p,p,p+1} - u_p v_1) - \mathcal{D}^{\mathcal{L}}(K_{p,p,p+1})$

$$= \begin{bmatrix} 0_{p-1\times p-1} & 2X' & X' & 0_{p-1\times p-1} & 0_{p-1\times p+1} \\ 2X'^t & 0 & x' & X'^t & Y'^t \\ X'^t & x' & 0 & 2X'^t & Y'^t \\ 0_{p-1\times p-1} & X' & 2X' & 0_{p-1\times p-1} & 0_{p-1\times p+1} \\ 0_{p+1\times p-1} & Y' & Y' & 0_{p+1\times p-1} & 0_{p+1\times p+1} \end{bmatrix}$$

where $x' = \frac{-2}{4p}+\frac{1}{4p-1}$, $X' = (\frac{-1}{\sqrt{4p(4p-1)}}+\frac{1}{4p-1})J_{p-1\times 1}$ and $Y' = (\frac{-1}{4p}+\frac{1}{\sqrt{4p(4p-1)}})J_{p+1\times 1}$. Since the matrix $M'$ here is similar to the difference of normalized distance Laplacian matrices of $K_{p,p,\ldots,p}$ and $K_{p,p,\cdot,p}-e$. Thus, the spectrum of $M'$ contains the eigenvalue 0 with multiplicity $3p-3$ and the remaining four eigenvalues, each with multiplicity 1, are given by:

$$\eta_1, \eta_2 = \frac{-x' \pm \sqrt{x'^2 + 4\|X'\|^2}}{2} \quad \text{and} \quad \eta_3, \eta_4 = \frac{x' \pm \sqrt{x'^2 + 4(9\|X'\|^2 + 2\|Y'\|^2)}}{2}.$$

We have,

$$|\eta_1| + |\eta_2| + |\eta_3| + |\eta_4| = \sqrt{x'^2 + 4||X'||^2} + \sqrt{x'^2 + 4(9||X'||^2 + 2||Y'||^2)}$$

where

$$x' = \frac{-2}{4p}+\frac{1}{4p-1},\ X' = (\frac{-1}{\sqrt{4p(4p-1)}}+\frac{1}{4p-1})J_{p-1\times 1} \text{ and } Y' = (\frac{-1}{4p}+\frac{1}{\sqrt{4p(4p-1)}})J_{p+1\times 1}$$

Now, we have, $x' = \frac{-2}{4p}+\frac{1}{4p-1}$, $X' = (\frac{-1}{\sqrt{4p(4p-1)}}+\frac{1}{4p-1})J_{p-1\times 1}$ and $Y' = (\frac{-1}{4p}+\frac{1}{\sqrt{4p(4p-1)}})J_{p+1\times 1}$.

$\frac{p-1}{4p-1} \leq \frac{1}{4}$ as $p$ tends to $\infty$ and $\frac{-1}{\sqrt{4p(4p-1)}} + \frac{1}{4p-1} \leq 0.023$ implies

$$|x'| \leq \frac{3}{28}$$

$$||X'||^2 = (p-1)(\frac{-1}{\sqrt{4p(4p-1)}} + \frac{1}{4p-1})^2 \leq \frac{1}{4}(0.023)^2 \approx 0.000132$$

$$||Y'||^2 = (p+1)(\frac{-1}{4p} + \frac{1}{\sqrt{4p(4p-1)}})^2 \leq \frac{p+1}{4p}(\frac{-1}{4p} + \frac{1}{\sqrt{4p-1}})^2 \leq \frac{1}{4}(0.023)^2 \approx 0.000132$$

$$\begin{aligned}|\mathcal{D}^{\mathcal{L}}E(K_{p,p,p+1}) - \mathcal{D}^{\mathcal{L}}E(K_{p,p,p+1} - u_pv_1)| &\leq \sum_{i=1}^{n} \sigma_i(M) \\ &= \eta_1(M') + \eta_2(M') + \eta_3(M') + \eta_4(M') \\ &= \sqrt{x'^2 + 4\|X'\|^2} + \sqrt{x'^2 + 4(9\|X'\|^2 + 2\|Y'\|^2)} \\ &\leq 0.241056\end{aligned}$$

□

Next, let us consider the case where the two endpoints of the deleted edge $e$ belong to two vertex subsets with different cardinalities.

**Theorem 4.2.**

$$|\mathcal{D}^{\mathcal{L}}E(K_{p,p,p+1}) - \mathcal{D}^{\mathcal{L}}E(K_{p,p,p+1} - v_iw_j)| \leq \sqrt{z^2 + 2(A-B)} + \sqrt{z^2 + 2(A+B)}$$

*where the endpoints $v_i$ and $w_j$ of the deleted edge belong to vertex partite sets with different cardinalities,*
$A = ||P||^2+||Q||^2+||R||^2+||S||^2+||T||^2+||W||^2$ *and* $B = Q^tP+T^tS+W^tR+P^tQ+S^tT+R^tW$ *for*

$$\begin{aligned} P &= (\tfrac{-1}{\sqrt{4p(4p-1)}} + \tfrac{1}{4p-1})J_{p\times 1} \\ Q &= (\tfrac{-1}{\sqrt{(4p-1)(4p+1)}} + \tfrac{1}{\sqrt{4p(4p-1)}})J_{p\times 1} \\ R &= (\tfrac{-1}{4p} + \tfrac{1}{\sqrt{4p(4p-1)}})J_{p\times 1} \\ S &= (\tfrac{-2}{\sqrt{4p(4p-1)}} + \tfrac{2}{4p-1})J_{p-1\times 1} \\ T &= (\tfrac{-1}{\sqrt{(4p-1)(4p+1)}} + \tfrac{1}{\sqrt{4p(4p-1)}})J_{p-1\times 1} \\ W &= (\tfrac{-2}{\sqrt{4p(4p+1)}} + \tfrac{2}{4p})J_{p\times 1} \\ z &= \tfrac{-2}{\sqrt{4p(4p+1)}} + \tfrac{1}{\sqrt{4p(4p-1)}} \end{aligned}$$

*Proof.* Consider the graph $K_{p,p,p+1}$ with $V(K_{p,p,p+1}) = V_1 \cup V_2 \cup V_3 \cup V_4 \cup V_5$ and $p \geq 2$, where

$$U_1' = \{u_1, u_2, \ldots, u_p\}, \quad U_2' = \{v_1, v_2, \cdots, v_{p-1}\} \quad U_3' = \{v_p\}, \quad U_4' = \{w_1\} \quad U_5' = \{w_2, \ldots, w_{p+1}\}.$$

Let $v_p$ and $w_1$ be the end vertices of the edge $e$. In $K_{p,p,p+1}$, $Tr(v) = 4p-1$ for $v \in U_1', U_2', U_3'$ and $Tr(v) = 4p$ for $v \in U_4', U_5'$. Also, In $K_{p,p,p+1} - v_pw_1$, $Tr(v) = 4p-1$ for $v \in U_1', U_2'$, $Tr(v) = 4p$ for $v \in U_3', U_5'$ and $Tr(v) = 4p+1$ for $v \in U_4'$.
Hence the normalized distance Laplacian matrix of $K_{p,p,p+1}$ and $K_{p,p,p+1} - v_pw_1$ is given by

$$\mathcal{D}^{\mathcal{L}}(K_{p,p,p+1}) = \begin{bmatrix} \frac{-2(J-I)}{4p-1}+I & \frac{-1}{4p-1}J_{p\times p-1} & \frac{-1}{4p-1}J_{p\times 1} & \frac{-1}{\sqrt{4p(4p-1)}}J_{p\times 1} & \frac{-1}{\sqrt{4p(4p-1)}}J_{p\times p} \\ \frac{-1}{4p-1}J_{p-1\times p} & \frac{-2(J-I)}{4p-1}+I & \frac{-2}{4p-1}J & \frac{-1}{\sqrt{4p(4p-1)}}J & \frac{-1}{\sqrt{4p(4p-1)}}J \\ \frac{-1}{4p-1}J_{1\times p} & \frac{-2}{4p-1}J & 1 & \frac{-1}{\sqrt{4p(4p-1)}} & \frac{-1}{\sqrt{4p(4p-1)}}J \\ \frac{-1}{\sqrt{4p(4p-1)}}J_{1\times p} & \frac{-1}{\sqrt{4p(4p-1)}}J & \frac{-1}{\sqrt{4p(4p-1)}} & 1 & \frac{-2}{4p}J \\ \frac{-1}{\sqrt{4p(4p-1)}}J_{p\times p} & \frac{-1}{\sqrt{4p(4p-1)}}J & \frac{-1}{\sqrt{4p(4p-1)}}J & \frac{-2}{4p}J & \frac{-2(J-I)}{4p}+I \end{bmatrix}$$

$\mathcal{D}^{\mathcal{L}}(K_{p,p,p+1} - v_p w_1) =$

$$\begin{bmatrix} \frac{-2(J-I)}{4p-1}+I & \frac{-1}{4p-1}J_{p\times p-1} & \frac{-1}{\sqrt{4p(4p-1)}}J_{p\times 1} & \frac{-1}{\sqrt{(4p-1)(4p+1)}}J_{p\times 1} & \frac{-1}{\sqrt{4p(4p-1)}}J_{p\times p} \\ \frac{-1}{4p-1}J & \frac{-2(J-I)}{4p-1}+I & \frac{-2}{\sqrt{4p(4p-1)}}J & \frac{-1}{\sqrt{(4p-1)(4p+1)}}J & \frac{-1}{\sqrt{4p(4p-1)}}J \\ \frac{-1}{\sqrt{4p(4p-1)}}J & \frac{-2}{\sqrt{4p(4p-1)}}J & 1 & \frac{-2}{\sqrt{4p(4p+1)}} & \frac{-1}{4p}J \\ \frac{-1}{\sqrt{(4p-1)(4p+1)}}J & \frac{-1}{\sqrt{(4p-1)(4p+1)}}J & \frac{-2}{\sqrt{4p(4p+1)}} & 1 & \frac{-2}{\sqrt{4p(4p+1)}}J \\ \frac{-1}{\sqrt{4p(4p-1)}}J & \frac{-1}{\sqrt{4p(4p-1)}}J & \frac{-1}{4p}J & \frac{-2}{\sqrt{4p(4p+1)}}J & \frac{-2(J-I)}{4p}+I \end{bmatrix}$$

Hence

$$\begin{aligned} N' &= \mathcal{D}^{\mathcal{L}}(K_{p,p,p+1} - v_p w_1) - \mathcal{D}^{\mathcal{L}}(K_{p,p,p+1}) \\ &= \begin{bmatrix} 0_{p\times p} & 0_{p\times p-1} & P_{p\times 1} & Q_{p\times 1} & 0_{p\times p} \\ 0_{p-1\times p} & 0_{p-1\times p-1} & S_{p-1\times 1} & T_{p-1\times 1} & 0_{p-1\times p} \\ P^t & S^t & 0 & z & R^t \\ Q^t & T^t & z & 0 & W^t \\ 0_{p\times p} & 0_{p\times p-1} & R_{p\times 1} & W_{p\times 1} & 0_{p\times p} \end{bmatrix} \end{aligned}$$

where

$$\begin{aligned} P &= \left(\frac{-1}{\sqrt{4p(4p-1)}} + \frac{1}{4p-1}\right)J_{p\times 1} \\ Q &= \left(\frac{-1}{\sqrt{(4p-1)(4p+1)}} + \frac{1}{\sqrt{4p(4p-1)}}\right)J_{p\times 1} \\ R &= \left(\frac{-1}{4p} + \frac{1}{\sqrt{4p(4p-1)}}\right)J_{p\times 1} \\ S &= \left(\frac{-2}{\sqrt{4p(4p-1)}} + \frac{2}{4p-1}\right)J_{p-1\times 1} \\ T &= \left(\frac{-1}{\sqrt{(4p-1)(4p+1)}} + \frac{1}{\sqrt{4p(4p-1)}}\right)J_{p-1\times 1} \\ W &= \left(\frac{-2}{\sqrt{4p(4p+1)}} + \frac{2}{4p}\right)J_{p\times 1} \\ z &= \frac{-2}{\sqrt{4p(4p+1)}} + \frac{1}{\sqrt{4p(4p-1)}} \end{aligned}$$

Since $rank(N') = 4$, 0 is an eigenvalue of $N'$ with multiplicity $3p-3$.

For $\nu \neq 0$, $N' \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \\ a \\ b \\ \mathbf{w} \end{pmatrix} = \nu \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \\ a \\ b \\ \mathbf{w} \end{pmatrix}$ where $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are respectively $p\times 1$, $p-1\times 1$ and $p\times 1$

vectors gives the following equations,

$$aP + bQ = \nu\mathbf{u} \tag{4.1}$$
$$aS + bT = \nu\mathbf{v} \tag{4.2}$$
$$P^t\mathbf{u} + S^t\mathbf{v} + zb + R^t\mathbf{w} = \nu a \tag{4.3}$$
$$Q^t\mathbf{u} + T^t\mathbf{v} + za + W^T\mathbf{w} = \nu b \tag{4.4}$$
$$aR + bW = \nu\mathbf{w} \tag{4.5}$$

Substituting $\mathbf{u} = \frac{aP+bQ}{\nu}$, $\mathbf{v} = \frac{aS+bT}{\nu}$ and $\mathbf{w} = \frac{aR+bW}{\nu}$ in equations (4.3) and (4.4), we have

$$a(||P||^2 + ||S||^2 + ||R||^2 - \nu^2) + b(P^tQ + S^tT + \nu z + R^tW) = 0$$
$$a(Q^tP + T^tS + \nu z + W^tR) + b(||Q||^2 + ||T||^2 + ||W||^2 - \nu^2) = 0$$

Therefore $a \neq 0$ gives a four degree equation in $\nu$ whose solutions are eigenvalues of $N'$.

$$\nu^4 - \nu^2[||P||^2+||Q||^2+||R||^2+||S||^2+||T||^2+||W||^2+z^2] - \nu z[Q^tP+T^tS+W^tR+P^tQ+T^tS+R^tW]$$
$$+[(||P||^2+||Q||^2+||R||^2)(||Q||^2+||T||^2+||W||^2) - (Q^tP+T^tS+W^tR)(P^tQ+S^tT+R^tW)] = 0$$

This fourth degree equation can be factorised into

$$[\nu^2 + z\nu - \frac{A-B}{2}][\nu^2 - z\nu - \frac{A+B}{2}] = 0$$

where $A = ||P||^2 + ||Q||^2 + ||R||^2 + ||S||^2 + ||T||^2 + ||W||^2$ and $B = Q^tP + T^tS + W^tR + P^tQ + S^tT + R^tW$.

The solutions of the equation $\nu^2 + z\nu - \frac{A-B}{2} = 0$ is

$$\frac{-z \pm \sqrt{z^2 + 2(A-B)}}{2}$$

Similarly, the solutions of the equation $\nu^2 - z\nu - \frac{A+B}{2} = 0$ are

$$\frac{z \pm \sqrt{z^2 + 2(A+B)}}{2}$$

$A - B \geq 0$ and $A + B \geq 0$ implies $|-z| \leq |\sqrt{z^2 + 2(A-B)}|$ and $|z| \leq |\sqrt{z^2 + 2(A+B)}$.

Hence $\nu_1 = \frac{-z+\sqrt{z^2+2(A-B)}}{2} \leq 0$, $\nu_2 = \frac{-z-\sqrt{z^2+2(A-B)}}{2} \geq 0$, $\nu_3 = \frac{z+\sqrt{z^2+2(A+B)}}{2} \leq 0$ and $\nu_4 = \frac{z-\sqrt{z^2+2(A+B)}}{2} \geq 0$.

Therefore

$$|\nu_1| + |\nu_2| + |\nu_3| + |\nu_4| = \sqrt{z^2 + 2(A-B)} + \sqrt{z^2 + 2(A+B)}$$

$$\begin{aligned}|\mathcal{D}^{\mathcal{L}}E(K_{p,p,p+1}) - \mathcal{D}^{\mathcal{L}}E(K_{p,p,p+1} - v_iw_j)| &\leq \textstyle\sum_{i=1}^{n} \sigma_i(N') \\ &= |\nu_1(N')| + |\nu_2(N')| + |\nu_3(N')| + |\nu_4(N')| \\ &= \sqrt{z^2 + 2(A-B)} + \sqrt{z^2 + 2(A+B)}\end{aligned}$$

□

## 5. $\mathcal{D}^{\mathcal{L}}E$ CHANGE IN $K_{m,n}$ DUE TO AN EDGE DELETION

**Theorem 5.1.** *Consider the graph $K_{m,n}$ and $e$ be any edge. Then,*

$$|\mathcal{D}^{\mathcal{L}}E(K_{m,n}-e)-\mathcal{D}^{\mathcal{L}}E(K_{m,n})|$$
$$\leq \sqrt{z'^2+2[(\theta_1-\gamma_1)^2||P'||^2+(\theta_2-\gamma_2)^2||Q'||^2]}+\sqrt{z'^2+2[(\theta_1+\gamma_1)^2||P'||^2+(\theta_2+\gamma_2)^2||Q'||^2]}$$

*where*

$$\alpha'=n+2(m-1) \qquad \beta'=m+2(n-1)$$
$$P'=J_{(m-1)\times 1} \qquad Q'=J_{(n-1)\times 1}$$
$$\theta_1=\frac{-2}{\sqrt{\alpha'(\alpha'+2)}}+\frac{2}{\alpha'} \qquad \gamma_1=\frac{-1}{\sqrt{\alpha'(\beta'+2)}}+\frac{1}{\sqrt{\alpha'\beta'}}$$
$$\theta_2=\frac{-1}{\sqrt{(\alpha'+2)\beta'}}+\frac{1}{\sqrt{\alpha'\beta'}} \qquad \gamma_2=\frac{-2}{\sqrt{\beta'(\beta'+2)}}+\frac{2}{\beta'}$$

*and* $z'=\frac{-3}{\sqrt{(\alpha'+2)(\beta'+2)}}+\frac{1}{\sqrt{\alpha'\beta'}}$.

*Proof.* Let $\{u_1,u_2,...,u_m\}$, $\{v_1,v_2,...,v_n\}$ be the 2 partite sets of $K_{m,n}$ and $e=u_mv_1$. Partition the vertex set of $K_{m,n}$ into 4 sets, $W_1=\{u_1,u_2,...,u_{m-1}\}$, $W_2=\{u_m\}$, $W_3=\{v_1\}$ and $W_4=\{v_2,...,v_n\}$. In $K_{m,n}$, $Tr(v)=n+2(m-1))$ for all $v\in W_1\bigcup W_2$ and $Tr(v)=m+2(n-1))$ for all $v\in W_3\bigcup W_4$. For $K_{m,n}-e$, transmission changes for $u_m$ and $v_1$. $Tr(u_m)=n+2m$ and $Tr(v_1)=2n+m$. Let $\alpha'=n+2(m-1)$, $\beta'=m+2(n-1)$ and hence the normalized distance Laplacian matrix of $K_{m,n}$ and $K_{m,n}-e$ is given by

$$\mathcal{D}^{\mathcal{L}}(K_{m,n})=\begin{bmatrix} \frac{2(J-I)}{\alpha'}+I & \frac{-2}{\alpha'}J_{m-1\times 1} & \frac{-1}{\sqrt{\alpha'\beta'}}J_{m-1\times 1} & \frac{-1}{\sqrt{\alpha'\beta'}}J_{m-1\times n-1} \\ \frac{-2}{\alpha'}J & 1 & \frac{-1}{\sqrt{\alpha'\beta'}} & \frac{-1}{\sqrt{\alpha'\beta'}}J \\ \frac{-1}{\sqrt{\alpha'\beta'}}J & \frac{-1}{\sqrt{\alpha'\beta'}} & 1 & \frac{-2}{\beta'}J \\ \frac{-1}{\sqrt{\alpha'\beta'}}J & \frac{-1}{\sqrt{\alpha'\beta'}}J & \frac{-2}{\beta'}J & \frac{2(J-I)}{\beta'}+I \end{bmatrix}$$

$$\mathcal{D}^{\mathcal{L}}(K_{m,n}-e)=\begin{bmatrix} \frac{2(J-I)}{\alpha'}+I & \frac{-2}{\sqrt{\alpha'(\alpha'+2)}}J_{m-1\times 1} & \frac{-1}{\sqrt{\alpha'(\beta'+2)}}J_{m-1\times 1} & \frac{-1}{\sqrt{\alpha'\beta'}}J_{m-1\times n-1} \\ \frac{-2}{\sqrt{\alpha'(\alpha'+2)}}J & 1 & \frac{-3}{\sqrt{(\alpha'+2)(\beta'+2)}} & \frac{-1}{\sqrt{(\alpha'+2)\beta'}}J \\ \frac{-1}{\sqrt{\alpha'(\beta'+2)}}J & \frac{-3}{\sqrt{(\alpha'+2)(\beta'+2)}} & 1 & \frac{-2}{\sqrt{\beta'(\beta'+2)}}J \\ \frac{-1}{\sqrt{\alpha'\beta'}}J & \frac{-1}{\sqrt{(\alpha'+2)\beta'}}J & \frac{-2}{\sqrt{\beta'(\beta'+2)}}J & \frac{2(J-I)}{\beta'}+I \end{bmatrix}$$

Hence $M''=\mathcal{D}^{\mathcal{L}}(K_{m,n}-e)-\mathcal{D}^{\mathcal{L}}(K_{m,n})=$

$$=\begin{bmatrix} 0_{m-1\times m-1} & \theta_1P' & \gamma_1P' & 0 \\ \theta_1P'^t & 0 & z' & \theta_2Q'^t \\ \gamma_1P'^t & z' & 0 & \gamma_2Q'^t \\ 0 & \theta_2Q' & \gamma_2Q' & 0 \end{bmatrix}$$

where $\theta_1=\frac{-2}{\sqrt{\alpha'(\alpha'+2)}}+\frac{2}{\alpha'}$, $\gamma_1=\frac{-1}{\sqrt{\alpha'(\beta'+2)}}+\frac{1}{\sqrt{\alpha'\beta'}}$, $\theta_2'=\frac{-1}{\sqrt{(\alpha'+2)\beta'}}+\frac{1}{\sqrt{\alpha'\beta'}}$, $\gamma_2=\frac{-2}{\sqrt{\beta'(\beta'+2)}}+\frac{2}{\beta'}$, $P'=J_{m-1\times 1}$, $Q'=J_{n-1\times 1}$ and $z'=\frac{-3}{\sqrt{(\alpha'+2)(\beta'+2)}}+\frac{1}{\sqrt{\alpha'\beta'}}$.
Since $\theta_1\neq\theta_2$ and $\gamma_1\neq\gamma_2$, $rank(M)=4$ and hence 0 is an eigenvalue of $M$ with multiplicity $mn-4$.

For $\zeta\neq 0$, $M''\begin{pmatrix}\mathbf{u}\\ a\\ b\\ \mathbf{v}\end{pmatrix}=\zeta\begin{pmatrix}\mathbf{u}\\ a\\ b\\ \mathbf{v}\end{pmatrix}$ where $a$, $b$ are scalars and $\mathbf{u}$, $\mathbf{v}$ are respectively $m-1\times 1$,

$n-1\times 1$ vectors gives the following equations,

$$\theta_1 aP' + \gamma_1 bP' = \zeta \mathbf{u} \tag{5.1}$$

$$\theta_1 P'^t u + bz' + \theta_2 Q'^t \mathbf{v} = \zeta a \tag{5.2}$$

$$\gamma_1 P'^t u + az' + \gamma_2 Q'^t v = \zeta b \tag{5.3}$$

$$\theta_2 aQ' + \gamma_2 bQ' = \zeta \mathbf{v} \tag{5.4}$$

Substituting $u = \frac{\theta_1 a+\gamma_1 b}{\zeta}P'$, $\mathbf{v} = \frac{\theta_2 a+\gamma_2 b}{\zeta}Q'$ in equations 5.2 and 5.3 gives the equation

$$a\left(\frac{\theta_1^2||P'||^2}{\zeta} + \frac{\theta_2^2||Q'||^2}{\zeta} - \zeta\right) + b\left(z' + \frac{\theta_1\gamma_1||P'||^2}{\zeta} + \frac{\theta_2\gamma_2||Q'||^2}{\zeta}\right) = 0$$

$$a\left(z' + \frac{\theta_1\gamma_1||P'||^2}{\zeta} + \frac{\theta_2\gamma_2||Q'||^2}{\zeta}\right) + b\left(\frac{\gamma_1^2||P'||^2}{\zeta} + \frac{\gamma_2^2||Q'||^2}{\zeta} - \zeta\right) = 0$$

Solving we will have a four degree equation in $\zeta$

$$\begin{aligned}&\zeta^4 - \zeta^2[z'^2 + (\theta_1^2+\gamma_1^2)\|P'\|^2 + (\theta_2^2+\gamma_2^2)\|Q'\|^2]\\ &\quad - 2\zeta z'[\theta_1\gamma_1\|P'\|^2 + \theta_2\gamma_2\|Q'\|^2] + \|P'\|^2\|Q'\|^2(\theta_1\gamma_2 - \theta_2\gamma_1)^2 = 0\end{aligned}$$

This can be factored into

$$\begin{aligned}&\left(\zeta^2 + \zeta z' - \frac{1}{2}[(\theta_1-\gamma_1)^2\|P'\|^2 + (\theta_2-\gamma_2)^2\|Q'\|^2]\right)\\ &\quad\times\left(\zeta^2 - \zeta z' - \frac{1}{2}[(\theta_1+\gamma_1)^2\|P'\|^2 + (\theta_2+\gamma_2)^2\|Q'\|^2]\right) = 0\end{aligned}$$

whose solutions are

$$\begin{aligned}\zeta_1 &= \frac{-z' + \sqrt{z'^2 + 2[(\theta_1-\gamma_1)^2|P'|^2 + (\theta_2-\gamma_2)^2|Q'|^2]}}{2},\\ \zeta_2 &= \frac{-z' - \sqrt{z'^2 + 2[(\theta_1-\gamma_1)^2|P'|^2 + (\theta_2-\gamma_2)^2|Q'|^2]}}{2},\\ \zeta_3 &= \frac{z' + \sqrt{z'^2 + 2[(\theta_1+\gamma_1)^2|P'|^2 + (\theta_2+\gamma_2)^2|Q'|^2]}}{2},\\ \zeta_4 &= \frac{z' - \sqrt{z'^2 + 2[(\theta_1+\gamma_1)^2|P'|^2 + (\theta_2+\gamma_2)^2|Q'|^2]}}{2},\end{aligned}$$

Consider the factor $(\zeta^2 + \zeta z' - \frac{1}{2}[(\theta_1-\gamma_1)^2||P'||^2 + (\theta_2-\gamma_2)^2||Q'||^2])$ whose solutions are

$$\frac{-z' \pm \sqrt{z'^2 + 2[(\theta_1-\gamma_1)^2||P'||^2 + (\theta_2-\gamma_2)^2||Q'||^2]}}{2}$$

Since $\sqrt{z'^2 + 2[(\theta_1-\gamma_1)^2||P'||^2 + (\theta_2-\gamma_2)^2||Q'||^2]} \geq z'$, we have

$$\zeta_1 = \frac{-z' + \sqrt{z'^2 + 2[(\theta_1-\gamma_1)^2||P'||^2 + (\theta_2-\gamma_2)^2||Q'||^2]}}{2} \geq 0$$

$$\zeta_2 = \frac{-z' - \sqrt{z'^2 + 2[(\theta_1-\gamma_1)^2||P'||^2 + (\theta_2-\gamma_2)^2||Q'||^2]}}{2} \leq 0$$

Therefore, $|\zeta_1| + |\zeta_2| = \sqrt{z'^2 + 2[(\theta_1-\gamma_1)^2||P'||^2 + (\theta_2-\gamma_2)^2||Q'||^2]}$.
Similarly considering the factor $(\zeta^2 - \zeta z' - \frac{1}{2}[(\theta_1+\gamma_1)^2||P'||^2 + (\theta_2+\gamma_2)^2||Q'||^2])$ whose solutions are

$$\frac{z' \pm \sqrt{z'^2 + 2[(\theta_1+\gamma_1)^2||P'||^2 + (\theta_2+\gamma_2)^2||Q'||^2]}}{2}$$

Since $\sqrt{z'^2 + 2[(\theta_1+\gamma_1)^2||P'||^2 + (\theta_2+\gamma_2)^2||Q'||^2]} \geq z'$, we have

$$\zeta_3 = \frac{-z' + \sqrt{z'^2 + 2[(\theta_1+\gamma_1)^2||P'||^2 + (\theta_2+\gamma_2)^2||Q'||^2]}}{2} \geq 0$$

$$\zeta_4 = \frac{-z' - \sqrt{z'^2 + 2[(\theta_1+\gamma_1)^2||P'||^2 + (\theta_2+\gamma_2)^2||Q'||^2]}}{2} \leq 0$$

Therefore, $|\zeta_3| + |\zeta_4| = \sqrt{z'^2 + 2[(\theta_1+\gamma_1)^2||P'||^2 + (\theta_2+\gamma_2)^2||Q'||^2]}$.

$$\begin{aligned}|\mathcal{D}^{\mathcal{L}}E(K_{m,n}) - \mathcal{D}^{\mathcal{L}}E(K_{m,n}-e)| &\leq |\zeta_1| + |\zeta_2| + |\zeta_3| + |\zeta_4| \\ &= \sqrt{z'^2 + 2[(\theta_1-\gamma_1)^2\|P'\|^2 + (\theta_2-\gamma_2)^2\|Q'\|^2]} \\ &\quad + \sqrt{z'^2 + 2[(\theta_1+\gamma_1)^2\|P'\|^2 + (\theta_2+\gamma_2)^2\|Q'\|^2]}\end{aligned}$$

where $\alpha' = n+2(m-1)$, $\beta' = m+2(n-1)$, $\theta_1 = \frac{-2}{\sqrt{\alpha'(\alpha'+2)}} + \frac{2}{\alpha'}$, $\gamma_1 = \frac{-1}{\sqrt{\alpha'(\beta'+2)}} + \frac{1}{\sqrt{\alpha'\beta'}}$, $\theta_2 = \frac{-1}{\sqrt{(\alpha'+2)\beta'}} + \frac{1}{\sqrt{\alpha'\beta'}}$, $\gamma_2 = \frac{-2}{\sqrt{\beta'(\beta'+2)}} + \frac{2}{\beta'}$, $P' = J_{m-1\times 1}$, $Q' = J_{n-1\times 1}$ and $z' = \frac{-3}{\sqrt{(\alpha'+2)(\beta'+2)}} + \frac{1}{\sqrt{\alpha'\beta'}}$.

□

## 6. Conclusions

In this study, we investigated how the normalized distance Laplacian energy changed by the deletion of an edge from $K_n$, $K_{p,p,\ldots,p}$, $K_{p,p,p+1}$, and $K_{m,n}$. Our findings suggest that the $\mathcal{D}^{\mathcal{L}}$-energy can either increase or decrease as a result of edge deletion. A similar analysis can be applied to the general case of $K_{n_1,n_2,\ldots,n_k}$.

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