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Research Paper

INTERSECTION GRAPH OF AMALGAMATED ALGEBRAS ALONG AN IDEAL

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ABSTRACT

Let $f : R \to S$ be a ring homomorphism of commutative rings with identity, and let J be a non-zero proper ideal of S. The amalgamation of R with S along J with respect to f denoted by $R \bowtie^f J$, was introduced by D'Anna et al. in 2010. In this paper, we investigate some properties of the intersection graph of ideals of $R \bowtie^f J$. We show that $\Gamma(R \bowtie^f J)$ is always connected and $\mathrm{diam}(\Gamma(R \bowtie^f J)) \leq 2$. We obtain some conditions which implies that for an integer $n > 0$, K_n is a subgraph of $\Gamma(R \bowtie^f J)$. We show that if R is a local ring, and $J \subseteq \mathrm{J}(S)$, where $\mathrm{J}(S)$ is the Jacobson radical of S, then $\Gamma(R \bowtie^f J)$ is planar if and only if $\Gamma(R \bowtie^f J)$ is star graph or K_3 or K_4, provided under certain conditions. Finally, we study the dominating number of $\Gamma(R \bowtie^f J)$.

1. Introduction

Throughout this paper all rings are considered commutative with identity element. There are many ways to associate a graph with an algebraic structure. The study of algebraic

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structures by way of graph theory is an exciting research topics. By associating a graph with an algebraic structure, we get representations of special classes of algebraic structures in terms of graphs and vice versa. Various graphs over algebraic structures were defined related to the intersection graphs, see [4], [5], [9], [10], and [18]. There are many papers on assigning a graph to a ring such as the zero-divisor graph, the total graph, the annihilating-ideal graph, and the comaximal graph. From now on let R be a ring. The intersection graph of ideals of R was introduced in [5]. Many authors have studied some properties of the ring R by studying the graph theoretics concepts of the intersection graph of ideals of R. The intersection graph of ideals of R, $\Gamma(R)$, is the undirected simple graph whose vertices are in a one to one correspondence with all non-zero proper ideals of R. The distinct vertices of $\Gamma(R)$ are adjacent if and only if the corresponding ideals of R have a non-zero intersection, see [5].

In this paper, we investigate some properties of the intersection graph of ideals of the extension of the ring R called amalgamation. Let $f : R \to S$ be a ring homomorphism, and let J be an ideal of S. The amalgamation of R with S along J with respect to f denoted by $R \bowtie^f J$, introduced in [12], is the following subring of $R \times S$:

$$R \bowtie^f J = \{(r, f(r) + j) \mid r \in R, j \in J\}.$$

This construction is a generalization of the amalgamated duplication of a ring along an ideal introduced in [15], and has been studied in [6], [7], [11], [16], [23], [24], and [25]. Several properties of the construction $R \bowtie^f J$ are investigated in [8], [12], [14], [21], and [27]. The diameter of the zero-divisor graph of an amalgamated algebra is studied in [3]. Also, some properties of the comaximal graph of an amalgamation algebra are studied in [26].

In [1], [2], [5], and [20] some properties of the intersection graph of ideals of commutative rings are investigated. In this paper, we deal with some properties of the intersection graph of ideals of R which are transferred to the intersection graph of ideals of $R \bowtie^f J$ and vice versa. It is shown that $\Gamma(R)$ is a subgraph of $\Gamma(R \bowtie^f J)$, see Proposition 2.2. In Theorem 2.9, we obtain some conditions which implies that for an integer $n > 0$, K_n is a subgraph of $\Gamma(R \bowtie^f J)$. Also, we study the planetary of $\Gamma(R \bowtie^f J)$. It is investigated that $\Gamma(R \bowtie I)$ is not a planar graph provided that I, H, and K are non-zero proper ideals of R such that $I \nsubseteq H$, $H \nsubseteq K$, and $H \cap K =< 0 >$, see Theorem 2.12. In addition, in Proposition 2.14 it is shown that if R is a local ring and $J \subseteq \mathrm{J}(S)$, where $\mathrm{J}(S)$ is the Jacobson radical of S, then $\Gamma(R \bowtie^f J)$ is planar if and only if $\Gamma(R \bowtie^f J)$ is star graph or K_3 or K_4, provided that there exists a non-zero proper ideal of R, I such that $f(I) \subseteq J$. It is shown that $\Gamma(R \bowtie^f J)$ is always connected and $\mathrm{diam}(\Gamma(R \bowtie^f J)) \leq 2$, see Lemma 2.25. Also, we get some results about completeness of $\Gamma(R \bowtie^f J)$. Over Artinian local ring R, it is shown that $\Gamma(R \bowtie^f J)$ is complete if and only if $f(R) + J$ is a Cohen-Macaulay ring, provided some special conditions, see Theorem 2.22. Finally, we study the dominating number of $\Gamma(R \bowtie^f J)$. In Theorem 2.31, it is shown that if R is an Artinian ring which is not a field, then the dominating number of $\Gamma(R)$ is equal 2 if and only if the dominating number of $\Gamma(R \bowtie^f J)$ is equal 2, provided under certain conditions.

2. Main Results

Let $f : R \to S$ be a ring homomorphism, and let $J \neq <0>$ be a proper ideal of S. The *amalgamation of* R *with* S *along* J *with respect to* f denoted by $R \bowtie^f J$, is the following subring of $R \times S$:

$$R \bowtie^f J = \{(r, f(r) + j) \mid r \in R, j \in J\}.$$

This notion is introduced in [14]. In the case $S = R$, we can consider the identity map $\mathrm{id} := \mathrm{id}_R : R \to R$, and construct $R \bowtie^{\mathrm{id}} J$. This construction is also called *amalgamation of* R *along* J instead of amalgamation of R with R along J with respect to id. Also, we use notation $R \bowtie J$ instead of $R \bowtie^{\mathrm{id}} J$. In this section, we study some properties of intersection graph of ideals of $R \bowtie^f J$. The intersection graph of ideals of R is introduced in [5], as the following.

Definition 2.1. The intersection graph of ideals of R is the undirected simple graph whose vertices are in a one to one correspondence with all non-zero proper ideals of R. The distinct vertices of this graph are adjacent if and only if the corresponding ideals of R have a non-zero intersection.

We denote the intersection graph of ideals of R by $\Gamma(R)$. Throughout this section, let $f : R \to S$ be a ring homomorphism, and let $<0> \neq J$ be a proper ideal of S. We follow standard notation and terminology about graph theory from [17].

Proposition 2.2. *For the ring* R*,* $\Gamma(R)$ *is a subgraph of* $\Gamma(R \bowtie^f J)$*.*

Proof. Let $<0> \neq I$ be a proper ideal of R. Then $<0> \neq I \bowtie^f J$ is a proper ideal of $R \bowtie^f J$ by [12, Proposition 5.1]. Therefore, $V(\Gamma(R)) \subseteq V(\Gamma(R \bowtie^f J))$. Suppose that $I_1 \neq <0>$ and $I_2 \neq <0>$ are proper ideals of R such that $I_1 \cap I_2 \neq <0>$. Then, there exists $0 \neq r \in I_1 \cap I_2$ and so $(0, 0) \neq (r, f(r)) \in (I_1 \bowtie^f J) \cap (I_2 \bowtie^f J)$. Hence, $E(\Gamma(R)) \subseteq E(\Gamma(R \bowtie^f J))$. □

The *girth* of a graph G denoted by $\mathrm{girth}(G)$, is the length of a shortest cycle contained in G.

Corollary 2.3. $\mathrm{girth}(\Gamma(R \bowtie^f J)) \leq \mathrm{girth}(\Gamma(R))$.

A *clique* of a graph is its maximal complete subgraph and the number of vertices in the largest clique of a graph G, denoted by $\mathrm{clique}(G)$, is called the *clique number* of G.

Corollary 2.4. $\mathrm{clique}(\Gamma(R)) \leq \mathrm{clique}(\Gamma(R \bowtie^f J))$.

Corollary 2.5. *Let* $\mathrm{clique}(\Gamma(R \bowtie^f J)) < \infty$*. Then* R *is a Artinian ring.*

Proof. By Corollary 2.4, $\mathrm{clique}(\Gamma(R)) < \infty$, and so R is Artinian by [2, Lemma 15]. □

Corollary 2.6. *Let* $<0> \neq I$ *be a proper ideal of* R*, and let* v_I *be a corresponding vertex of* I *in* $\Gamma(R)$ *and* $v_{I \bowtie^f J}$ *be a corresponding vertex of* $I \bowtie^f J$ *in* $\Gamma(R \bowtie^f J)$*. If* $\deg_{\Gamma(R)}(v_I) = n$*, then* $\deg_{\Gamma(R \bowtie^f J)}(v_{I \bowtie^f J}) \geq n$

Example 2.7. Let $f = id_{\mathbb{Z}_4} : \mathbb{Z}_4 \to \mathbb{Z}_4$ and let $J = \{0, 2\}$. Therefore, $\mathbb{Z}_4 \bowtie^f J$ equal $J = \{(0, 0), (0, 2), (1, 1), (1, 3), (2, 2), (2, 0), (3, 3), (3, 1)\}$. Therefore, the set of ideals of $\mathbb{Z}_4 \bowtie^f J$ is $\{<0>, I_1, I_2, I_3, \mathbb{Z}_4 \bowtie^f J\}$, where, $I_1 = \{(0, 0), (0, 2), (2, 2), (2, 0)\}$, $I_2 = \{(0, 0), (0, 2)\}$ and $I_3 = \{(0, 0), (2, 0)\}$. So, $I_1 \cap I_2 \neq <0>$, $I_1 \cap I_3 \neq <0>$ and $I_2 \cap I_3 = <0>$. Hence, $\Gamma(\mathbb{Z}_4 \bowtie^f J)$ is a star graph.

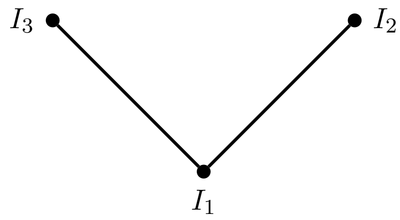

FIGURE 1.

In the following, we obtain some conditions which implies that for an integer $n > 0$, K_n is a subgraph of $\Gamma(R \bowtie^f J)$. First, we consider the following proposition about the ideals of $R \bowtie^f J$.

Proposition 2.8. *Let $< 0 >\neq I$ be a proper ideal of R, and let J' be an ideal of S such that $f(I) \subseteq J' \subseteq J$. Then the following statements hold.*
(i) *$I \bowtie^f J'$ is a non-zero proper ideal of $R \bowtie^f J$.*
(ii) *If $J' \neq J$, then $I \bowtie^f J' \neq I \bowtie^f J$.*
(iii) *$(I \bowtie^f J') \cap (I \bowtie^f J) \neq < 0 >_{R \bowtie^f J}$.*
(iv) *The ideal $I\times < 0 >$ is a non-zero proper ideal of $R \bowtie^f J$. In addition, $(I\times < 0 >) \cap (I \bowtie^f J) \neq < 0 >_{R \bowtie^f J}$, and $(I\times < 0 >) \cap (I \bowtie^f J') \neq < 0 >_{R \bowtie^f J}$.*

Proof. It is routine. □

Theorem 2.9. *Let $< 0 >\neq I$ be a proper ideal of R. Then the following statements hold.*
(i) Assume that $A = \{J' \mid J'$ is an ideal of S such that $f(I) \subseteq J' \subsetneqq J\}$ with $\mid A \mid= n$. If $f(I) \neq \{0\}$, then $K_{1,n+1}$ is a subgraph of $\Gamma(R \bowtie^f J)$. Otherwise, $K_{1,n}$ is a subgraph of $\Gamma(R \bowtie^f J)$.
(ii) Assume that $B = \{J_1 \subseteq J_2 \subseteq \cdots \subseteq J_n = J \mid J_i$ is an ideal of S for $i = 1, 2, \dots, n$, and $f(I) \subseteq J_1\}$. If $f(I) \neq \{0\}$, then K_{n+1} is a subgraph of $\Gamma(R \bowtie^f J)$. Otherwise, K_n is a subgraph of $\Gamma(R \bowtie^f J)$.

Proof. (i) Let $A = \{J_1, J_2, \dots, J_n\}$. Then $I \bowtie^f J_1, I \bowtie^f J_2, \dots, I \bowtie^f J_n \in V(\Gamma(R \bowtie^f J))$, by Proposition 2.8. Also, the vertices $I \bowtie^f J$ and $I \bowtie^f J_i$ are adjacent for $i = 1, 2, \dots, n$. On the other hand, if $f(I) \neq \{0\}$, then $I\times < 0 >\in V(\Gamma(R \bowtie^f J))$, and the vertices $I\times < 0 >$ and $I \bowtie^f J$ are adjacent. So, $K_{1,n+1}$ is a subgraph of $\Gamma(R \bowtie^f J)$.
(ii) Note that $I \bowtie^f J_1, I \bowtie^f J_2, \dots, I \bowtie^f J_n \in V(\Gamma(R \bowtie^f J))$, by Proposition 2.8. Also, the vertices $I \bowtie^f J_i$ and $I \bowtie^f J_j$ are adjacent for $i, j = 1, 2, \dots, n$. On the other hand, if $f(I) \neq \{0\}$, then $I\times < 0 >\in V(\Gamma(R \bowtie^f J))$, and the vertices $I\times < 0 >$ and $I \bowtie^f J_i$ are adjacent for $i = 1, 2, \dots, n$. Then K_{n+1} is a subgraph of $\Gamma(R \bowtie^f J)$. □

Example 2.10. Let $f = id_{\mathbb{Z}_8} : \mathbb{Z}_8 \to \mathbb{Z}_8$ and let $J = \{0, 2, 4, 6\}$. Then $< 0 >\subsetneqq \{0, 4\} \subsetneqq J$. Set $I_1 = \{0, 2, 4, 6\}$, set vertex v_1 corresponding to the ideal $I_1 \bowtie^f J$, the vertex v_2 corresponding to the ideal $I_1 \bowtie^f \{0, 4\}$ and the vertex v_3 corresponding to the ideal $I_1\times < 0 >$. Then v_1, v_2 and v_3 make K_3 in $\Gamma(\mathbb{Z}_8 \bowtie^f J)$, by Theorem 2.9. Set $I_2 = \{0, 4\}$, set the vertex v_4 corresponding to the ideal $I_2 \bowtie^f J$, the vertex v_5 corresponding to the ideal $I_2 \bowtie^f \{0, 4\}$ and the vertex v_6 corresponding to the ideal $I_2\times < 0 >$. Then v_4, v_5 and v_6 make K_3 and $\{v_1, v_2, \dots, v_6\}$ make K_6 in $\Gamma(\mathbb{Z}_8 \bowtie^f J)$, see Figures 2 and 3.

Proposition 2.11. *Let I and H be non-zero proper ideals of R such that $I \nsubseteq H$. Then $\Gamma(R \bowtie I)$ has a subgraph isomorphic to K_3. Furthermore, if $IH \neq I \cap H$, then $\Gamma(R \bowtie I)$ has a subgraph isomorphic to K_4.*

Proof. Set

$$\begin{aligned} H_0 &= \{(h, h+i) \mid h \in H,\ i \in I \cap H\}, \\ H_1 &= \{(h, h+i) \mid h \in H,\ i \in I\}, \\ H_2 &= \{(h+i, h) \mid h \in H,\ i \in I\}, \\ H^e &= \{(h, h+i) \mid h \in H,\ i \in IH\}. \end{aligned}$$

By [11, Proposition 5], H_0, H_1, H_2 and H^e are ideals of $R \bowtie I$. Since, $< 0 > \neq H$, there exists $0 \neq h \in H$ such that $(h, h) \neq (0, 0)$ and $(h, h) \in H_0, H_1, H_2, H^e$. Now, $I \nsubseteq H$, implies that $H_1 \neq H_2$ and $H_1 \cap H_2 = H_0$. In addition, if $IH \neq I \cap H$, then $H^e \neq H_0$. So, we get the result. □

Recall that a *planar graph* is a graph that can be embedded in the plane, i.e., it can be drawn on the plane in such a way that its edges intersect only at their vertices. In the following, we study the planetary of $\Gamma(R \bowtie^f J)$.

Theorem 2.12. *Let I, H and K be non-zero proper ideals of R such that $I \nsubseteq H$ and $H \nsubseteq K$. If $H \cap K \neq < 0 >$. Then $\Gamma(R \bowtie I)$ is not planar.*

Proof. Set

$$\begin{aligned} H_0 &= \{(h, h+i) \mid h \in H,\ i \in I \cap H\}, \\ H_1 &= \{(h, h+i) \mid h \in H,\ i \in I\}, \\ H_2 &= \{(h+i, h) \mid h \in H,\ i \in I\}, \\ K_0 &= \{(h, h+i) \mid h \in K,\ i \in I \cap K\}, \\ K_1 &= \{(h, h+i) \mid h \in K,\ i \in I\}, \\ K_2 &= \{(h+i, h) \mid h \in K,\ i \in I\}. \end{aligned}$$

By [11, Proposition 5], H_0, H_1, H_2, K_0, K_1 and K_2 are ideals of $R \bowtie I$. Since, $H \cap K \neq < 0 >$, there exists $0 \neq h \in H \cap K$ such that $(h, h) \neq (0, 0)$ and $(h, h) \in H_0, H_1, H_2, K_0, K_1, K_2$. Then $K_{3,3}$ is a subgraph of $\Gamma(R \bowtie I)$ and so $\Gamma(R \bowtie I)$ is not planar. □

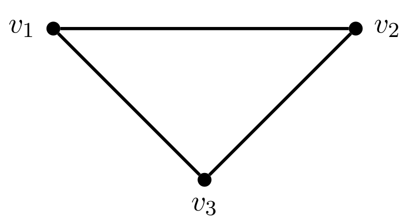

FIGURE 2.

Proposition 2.13. *Let $\Gamma(R \bowtie^f J)$ be planar, and $J \subseteq \mathrm{J}(S)$. Then $\mid \mathrm{Max}(R) \mid \leq 3$.*

Proof. By [26, Corollary 3.3], we have $\mid \mathrm{Max}(R) \mid = \mid \mathrm{Max}(R \bowtie^f J) \mid$. On the other hand, $\mid \mathrm{Max}(R \bowtie^f J) \mid \leq 3$, by [20, Lemma 2.6]. So, we get the assertion. □

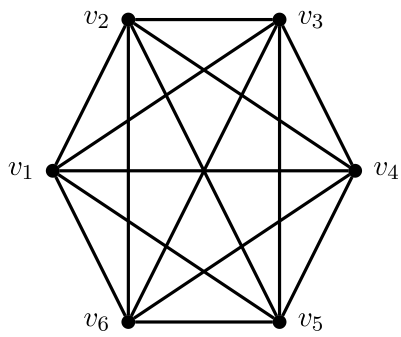

FIGURE 3.

Proposition 2.14. *Let $(R, \mathfrak{m})$ be a local ring, and let $J \subseteq \mathrm{J}(S)$. Assume that I is a non-zero proper ideal of R such that $f(I) \subseteq J$. Then $\Gamma(R \bowtie^f J)$ is planar if and only if $\Gamma(R \bowtie^f J)$ is star graph or K_3 or K_4.*

Proof. Note that $R \bowtie^f J$ is a local ring, by [13, Corollary 2.7]. Also, $I\times < 0 >$ and $I \bowtie^f J$ are non-zero proper ideals of $R \bowtie^f J$. So, $\Gamma(R \bowtie^f J)$ is not isomorphic to K_1. Hence, $\Gamma(R \bowtie^f J)$ is planar if and only if $\Gamma(R \bowtie^f J)$ is star graph or K_3 or K_4, by [20, Theorem 2.24]. □

Lemma 2.15. *Let $< 0 >\neq K \subseteq J$ be an ideal of $f(R) + J$. Then $< 0 > \times K$ is a non-zero proper ideal of $R \bowtie^f J$.*

Proof. Let k and k' be non-zero elements of K, and $(0, k), (0, k') \in < 0 > \times K$. Note that, $(0, k) + (0, k') = (0, k + k') \in < 0 > \times K$. Let $(r, f(r) + j) \in R \bowtie^f J$. By assumption $(0, k)(r, f(r)+j) = (0, k(f(r)+j))$, and $k(f(r)+j) \in K$. So, $< 0 > \times K$ is a non-zero proper ideal of $R \bowtie^f J$. □

Proposition 2.16. *Let $I_1, I_2, \ldots, I_n$ be ideals of $f(R)+J$ such that $< 0 >\subsetneqq I_1 \subsetneqq I_2 \subsetneqq \cdots \subsetneqq I_n \subseteq J$. Then K_n is a subgraph of $\Gamma(R \bowtie^f J)$. In addition, if $n \geq 5$, then $\Gamma(R \bowtie^f J)$ is not planar.*

Proof. By Lemma 2.15, $< 0 > \times I_i$ is a non-zero proper ideal of $R \bowtie^f J$, for $i = 1, 2, \ldots, n$. Since $I_1 \neq < 0 >$, there exists $0 \neq a \in I_1$, and so $(0, a) \in \bigcap_{i=1}^{n} (0 \times I_i)$. Therefore, K_n is a subgraph of $\Gamma(R \bowtie^f J)$, and the assertion follows from [20, Lemma 2.1]. □

Notation 2.17. Let L be an ideal of $R \bowtie^f J$. Set $L_J = \{r \in R \mid \exists j \in J$ such that $(r, f(r) + j) \in L\}$, which is an ideal of R. Note that $L \neq < 0 >_{R\bowtie^f J}$ does not imply that $L_J \neq < 0 >_R$, see I_2 in Example 2.7.

Proposition 2.18. *Let L and L' be non-zero proper ideals of $R \bowtie^f J$ such that $L_J \neq < 0 >$ and $L'_J \neq < 0 >$. Assume that v_1 and v_2 be corresponding vertices of L and L' in $\Gamma(R \bowtie^f J)$ and w_1 and w_2 be corresponding vertices of L_J and L'_J in $\Gamma(R)$. If $\mathrm{d}_{\Gamma(R\bowtie^f J)}(v_1, v_2) = 1$, then $\mathrm{d}_{\Gamma(R)}(w_1, w_2) = 1$, provided that for every $0 \neq j \in J$ we have $(0, j) \notin L \cap L'$.*

Proof. Note that $L \cap L' \neq < 0 >_{R\bowtie^f J}$, since $\mathrm{d}_{\Gamma(R\bowtie^f J)}(v_1, v_2) = 1$. Hence, there exists $(r, f(r) + j) \neq (0, 0)$ such that $(r, f(r) + j) \in L \cap L'$. If $r \neq 0$, then $L_J \cap L'_J \neq < 0 >$ and so $\mathrm{d}_{\Gamma(R)}(w_1, w_2) = 1$. If $r = 0$, then $(0, j) \in L \cap L'$ which is a contradiction. □

Theorem 2.19. *Let I_1 and I_2 be non-zero proper ideals of R such that $I_1 \subsetneqq f^{-1}(J)$ and $I_2 \subsetneqq f^{-1}(J)$. If $I_1 \cap I_2 \neq <0>$, then $\Gamma(R \bowtie^f J)$ has a subgraph isomorphic to K_4.*

Proof. Note that $I_1 \times <0>$ and $I_2 \times <0>$ are ideals of $R \bowtie^f J$ by Proposition 2.8. Also, the assumption $I_1 \cap I_2 \neq <0>$ implies that $(I_1 \times <0>) \cap (I_2 \times <0>) \neq <0>$, and $(I_1 \bowtie^f J) \cap (I_2 \bowtie^f J) \neq <0>$. Let $0 \neq r \in I_1 \cap I_2$. Then there exists $j \in J$ such that $r = f^{-1}(j)$, since $I_1 \cap I_2 \subseteq f^{-1}(J)$. Therefore, $f(r) \in J$ and so $(r, 0) = (r, f(r) - f(r)) \in (I_1 \bowtie^f J) \cap (I_2 \bowtie^f J) \cap (I_1 \times <0>) \cap (I_2 \times <0>)$. Hence, $\Gamma(R \bowtie^f J)$ has a subgraph isomorphic to K_4. □

Lemma 2.20. *Let R be a Noetherian ring, and let J be a finitely generated R-module. Assume that f is surjective. Then R is Artinian if and only if $R \bowtie^f J$ is Artinian.*

Proof. Let f be surjective. Then $\dim R = \dim(R \bowtie^f J)$, by [14, Proposition 4.1]. Also, $R \bowtie^f J$ is Noetherian by [12, Proposition 5.7], since J is a finitely generated R-module. So, we get the assertion. □

Corollary 2.21. *Let $<0> \neq I$ be a proper ideal of R. Then R is Artinian if and only if $R \bowtie I$ is Artinian.*

Recall that a finitely generated module M over a Noetherian ring R satisfies Serre's condition (S_n) if $\operatorname{depth} M_{\mathfrak{p}} = \min\{n, \dim M_{\mathfrak{p}}\}$, for all $\mathfrak{p} \in \operatorname{Spec}(R)$. Note that if M is Cohen-Macaulay, then it satisfies Serre's condition (S_n) for any integer n. Also, when $\dim M = d$ and M satisfies Serre's condition (S_d), then M is Cohen-Macaulay. Also, we recall that an element of a ring is *regular* if it is not a zero-divisor. An ideal is *regular* if it contains a regular element.

Theorem 2.22. *Let $(R, \mathfrak{m})$ be an Artinian local ring, and let f be surjective. Assume that $<0> \neq J \subseteq \mathrm{J}(S)$ and J is finitely generated R-module and $f(R) + J$ satisfies (S_1) and equidimensional. If $f^{-1}(J)$ is a regular ideal of R, then $\Gamma(R \bowtie^f J)$ is complete if and only if $f(R) + J$ is a Cohen-Macaulay ring, J is a canonical module of $f(R) + J$, and $f^{-1}(J)$ is a canonical module of R.*

Proof. Note that $(R, \mathfrak{m})$ is a Noetherian Cohen-Macaulay ring. By Lemma 2.20, $R \bowtie^f J$ is Artinian ring. Since $J \subseteq \mathrm{J}(S)$, we get that $R \bowtie^f J$ is local, by [13, Corollary 2.7]. Hence, $\Gamma(R \bowtie^f J)$ is complete if and only if $R \bowtie^f J$ is a Gorenstein ring, by [2, Theorem 14]. Therefore, we get the result, by [13, Proposition 5.7]. □

Theorem 2.23. *Let $(R, \mathfrak{m})$ be an Artinian local ring, and let $<0> \neq I$ be a proper ideal of R such that $\operatorname{Ann}_R(I) = <0>$. Then $\Gamma(R \bowtie I)$ is complete if and only if R has a canonical ideal w_R such that $I \cong w_R$.*

Proof. Note that $(R, \mathfrak{m})$ is a Noetherian Cohen-Macaulay ring. By Corollary 2.21, $R \bowtie I$ is Artinian ring. Also, $R \bowtie I$ is a local ring by [11, Corollary 6]. Hence, $\Gamma(R \bowtie I)$ is complete if and only if $R \bowtie I$ is Gorenstein by [2, Theorem 14]. Therefore, we get the assertion, by [11, Theorem 11]. □

In the following, we show that the graph $\Gamma(R \bowtie^f J)$ is always connected. First, we consider the following proposition.

Proposition 2.24. *The ring $R \bowtie^f J$ is not isomorphic of direct products of two fields.*

Proof. Suppose that there exists two fields F_1 and F_2 such that $R \bowtie^f J \cong F_1 \times F_2$. Note that $< 0 > \bowtie^f J$ is a non-zero proper ideal of $R \bowtie^f J$. So, $< 0 > \bowtie^f J \cong < 0 > \times F_2$ or $< 0 > \bowtie^f J \cong F_1 \times < 0 >$, which is a contradiction. □

The *diameter* of a connected graph is the supremum of the distances between vertices. The diameter of the graph G is denoted by $\mathrm{diam}(G)$.

Lemma 2.25. *The graph* $\Gamma(R \bowtie^f J)$ *is always connected and* $\mathrm{diam}(\Gamma(R \bowtie^f J)) \leq 2$.

Proof. It follows from Proposition 2.24, [5, Corollary 2.8] and [2, Theorem 4]. □

Proposition 2.26. *Let R be an Artinian ring, and let J be a finitely generated R-module. Assume that f is surjective. Then $R \bowtie^f J$ has a unique minimal ideal.*

Proof. Note that $\Gamma(R \bowtie^f J)$ is connected by Lemma 2.25 and $R \bowtie^f J$ is an Artinian ring by Lemma 2.20. Then we get the assertion, by [5, Theorem 2.11]. □

The ring $R \bowtie^f J$ satisfies the property $(*)$ if every ideal of $R \bowtie^f J$ has one of the following three forms:
(i) $I \times < 0 >$, where $I \subseteq f^{-1}(J)$ is an ideal of R.
(ii) $< 0 > \times K$, where $K \subseteq J$ is an ideal of $f(R) + J$.
(iii) $I \bowtie^f J$, where I is an ideal of R.

Recall that in [2, Theorem 11], it is shown that the ring R is an integral domain if and only if R is a reduced ring and $\Gamma(R)$ is a complete graph. Using this fact we get the following results.

Proposition 2.27. *Let f be injective, and assume that $R \bowtie^f J$ satisfies property $(*)$. Then $\Gamma(R)$ is complete.*

Proof. By [22, Theorem 2.1], R is an integral domain. Hence, $\Gamma(R)$ is complete by [2, Theorem 11]. □

Proposition 2.28. *Let S be an integral domain, and let $f^{-1}(J) = \{0\}$. Assume that R is reduced and $\Gamma(R)$ is complete. Then $\Gamma(R \bowtie^f J)$ is complete.*

Proof. By [2, Theorem 11], R is integral domain. Therefore, $R \bowtie^f J$ is integral domain by [12, Proposition 5.2]. So, we get the result by [2, Theorem 11]. □

For a graph $G = (V, E)$, a *dominating set* D is a subset of V such that every vertex not in D is adjacent to at least one vertex of D. The *domination number* $\gamma(G)$, is the minimum cardinality among all dominating sets of $G = (V, E)$. By [19], $\gamma(\Gamma(F)) = 0$, where F is a field, and $\gamma(\Gamma(R)) \leq 2$, by [1, Theorem 1].

Proposition 2.29. *Let S be an integral domain, and let $f^{-1}(J) = \{0\}$. Assume that R is reduced, and $\Gamma(R)$ is complete. Then $R \bowtie^f J$ is reduced.*

Proof. By [2, Theorem 11], R is integral domain. Therefore, $R \bowtie^f J$ is integral domain by [12, Proposition 5.2]. So, we get the result by [2, Theorem 11]. □

Lemma 2.30. *Assume that R is not a field. Then $R \bowtie^f J$ is not a field.*

Proof. By assumption, the set of non-zero proper ideal of R is not empty. Assume that I is a non-zero proper ideal of R. Then $I \bowtie^f J$ is a non-zero proper ideal of $R \bowtie^f J$. So, $R \bowtie^f J$ is not a field. □

Theorem 2.31. *Let R be an Artinian ring which is not a field. Assume that J is a finitely generated R-module and f is surjective. Then $\gamma(\Gamma(R)) = 2$ if and only if $\gamma(\Gamma(R \bowtie^f J)) = 2$, provided that $f(R)$ is reduced.*

Proof. Note that $R \bowtie^f J$ is an Artinian ring which is not a field by Lemma 2.20 and Lemma 2.30. Let $\gamma(\Gamma(R)) = 2$, then R is reduced, by [1, Corollary 9]. Hence, $R \bowtie^f J$ is reduced, by [12, Proposition 5.4]. So, $\gamma(\Gamma(R \bowtie^f J)) = 2$, by [1, Corollary 9]. The converse is proved by the same method. □

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