



Research Paper

ON ENUMERATION OF *EL*-HYPERGROUPS OF ORDER 3 OBTAINED FROM ORDERED SEMIGROUPS

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ABSTRACT

The *EL*-hyperstructures are formed from (quasi) partially ordered (semi)groups through the application of the “Ends Lemma”. In this paper, we enumerate *EL*-hypergroups of order 3, which are derived from ordered semigroups of order 3, using procedures implemented in MATLAB. Our approach is straightforward and accessible, leading to the identification of 27 *EL*-hypergroups of order 3.

1. INTRODUCTION

Enumeration of finite relational and algebraic structures is a classical topic in combinatorics. We consider preorders (quasi-orders), semigroups, and semigroups equipped with a multiplicatively compatible preorder. The enumeration of preorders is closely related to counting finite topologies: the correspondence was clarified by Sharp [44], with further foundations for finite spaces by Stong [45], and asymptotic results by Kleitman and Rothschild [27, 28]. In semigroup theory, the foundational framework was established by Clifford and

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Preston [2]. The asymptotic enumeration of finite semigroups was studied by Forsythe [15] and later by Kleitman et al. [25]. Exact enumeration, however, has largely relied on computational methods; a notable achievement is the classification and counting of semigroups of order 10 by Distler et al. [10]. Although ordered semigroups were introduced by Conrad [6], a systematic enumeration of pairs (S, \preceq) with a preorder compatible with multiplication is still relatively undeveloped.

The relationship between hyperstructures and ordering has been extensively studied by mathematicians such as Nieminen, Vougiouklis, Corsini, Rosenberg, Davvaz, Chvalina, Krasner, Mittas, and Leoreanu. A specific class of hyperstructures determined by binary relations is known as *EL*-hyperstructures. This concept was first introduced by Chvalina [5] in his investigation of quasi-ordered sets and hypergroups. Further studies on ordered semigroups and ordered groups in connection with *EL*-hyperstructures were conducted by Rosenberg [43], Hoskova [24], Rackova [42], and Novak [35, 36, 37, 38, 39, 41].

Ghazavi et al. [16] introduced a new class of *EL*-hyperstructures known as EL^2 -hyperstructures. These hyperstructures are based on (partially) quasi-ordered (semi)hypergroups instead of (partially) quasi-ordered (semi)groups. Subsequently, Ghazavi et al. further generalized *EL*-hyperstructures to EL_n -hyperstructures and *EL*- Γ -hyperstructures, as detailed in [17] and [18], respectively. They also, analyzed and studied the concept of a Γ -ideal within a quasi-ordered Γ -semigroup and its corresponding *EL*- Γ -semihypergroup in [21]. Recent results on the enumeration of hyperstructures have been obtained by C. Massouros and G. Massouros in [29, 30], including new developments on the borderline between fields and hyperfields and an enumeration and classification of hyperfields of order 7.

As previously mentioned, hyperstructure theory has been extensively studied, investigated, and extended over the past five decades from various perspectives. However, fewer works have focused on enumerating these structures. If H represents the set of all hypergroupoids with three elements, then its cardinality is given by $|H| = 7^9 = 40,353,607$. Migliorato [31] determined, using computational methods, that there are 23,192 hypergroups of order 3. Nordo [34], employing a program written in PASCAL, calculated that there are 3,999 non-isomorphic hypergroups of the same order [46]. Ghazavi and Mirvakili [19] classified *EL*-hypergroups with 2 elements, and further enumerated and classified EL^2 -semihypergroups and EL^2 - H_v -groups with 2 elements [20]. Now, we aim to identify all *EL*-hypergroups with 3 elements among the 3,999 hypergroups of order 3.

In this paper, we classify *EL*-hypergroups of order 3 using a straightforward approach that employs two procedures in MATLAB. Initially, we identify all ordered semigroups of order 3. Subsequently, we enumerate the *EL*-hypergroups of order 3, identifying all such hypergroups up to isomorphism and up to equivalent.

2. PRELIMINARIES

In this section, some basic definitions and properties are recalled to be considered later.

A hypergroupoid is a pair (H, \circ) where H is a nonempty set and $\circ : H \times H \rightarrow P^*(H)$ is a binary hyperoperation. ($P^*(H)$ is the system of all nonempty subsets of H). A semihypergroup is an associative hypergroupoid, i.e. hypergroupoid satisfying the equality $a \circ (b \circ c) = (a \circ b) \circ c$ for every triad $a, b, c \in H$. If moreover the semihypergroup H satisfies $a \circ H = H = H \circ a$, for all $a \in H$, it is called a *hypergroup*. This condition is known

as *reproduction axiom*.

In the above definition, if A and B are two non-empty subsets of H and $x \in H$, then $x \circ A = \{x\} \circ A$, $A \circ x = A \circ \{x\}$ and $A \circ B = \bigcup_{a \in A, b \in B} a \circ b$. For a deeper insight into the basic hyperstructure theory see [7] and [9].

Since the theory of relations and ordered structures is dealt with ordered relations, we need to recall some definitions in this respect. Binary relation R is called *quasi order* if it is reflexive and transitive. Also, if the binary relation R is reflexive, transitive and anti symmetric, then it is known as a partially order relation. By a quasi (partially) ordered (semi)group, we mean a triple (S, \cdot, R) , where (S, \cdot) is a (semi)group and R is a quasi (partially) order relation on S such that for all $x, y, z \in S$ with the property xRy there holds $(x \cdot z)R(y \cdot z)$ and $(z \cdot x)R(z \cdot y)$. This property is known as monotone condition. Moreover, the notation $[x]_R$ used below stands for the set $\{s \in S; xRs\}$ and also $[A]_R = \bigcup_{x \in A} [x]_R$.

Similarly, $(x)_R = \{s \in S; sRx\}$ and $(A)_R = \bigcup_{x \in A} (x)_R$. The EL -hyperstructures or *Ends lemma* based hyperstructures are hyperstructures constructed from a quasi (partially) ordered (semi)groups using "Ends lemma". This concept was first introduced by Chvalina in 1995 [5]. In particular, Chvalina proved that:

Lemma 2.1. ([5], Theorem 1.3) *Let (S, \cdot, R) be a partially ordered semigroup. The binary hyperoperation $\circ : S \times S \rightarrow P^*(S)$ defined by $a \circ b = [a \cdot b]_R = \{x \in S, a \cdot bRx\}$ is associative. The semihypergroup (S, \circ) is commutative if and only if the semigroup (S, \cdot) is commutative.*

Theorem 2.2. ([5], Theorem 1.4) *Let (S, \cdot, R) be a partially ordered semigroup. The following conditions are equivalent:*

- I) *For any pair $(a, b) \in S^2$ there exists a pair $(c, c_1) \in S^2$ such that $(b \cdot c)Ra$ and $(c_1 \cdot b)Ra$.*
- II) *The associated semihypergroup (S, \circ) is a hypergroup.*

Remark 2.3. If (S, \cdot, R) is a partially ordered group, then if we take $c = b^{-1} \cdot a$ and $c' = a \cdot b^{-1}$, then condition II is valid. Therefore, if (S, \cdot, R) is a partially ordered group, then its associated hyperstructure is a hypergroup.

Remark 2.4. The wording of the above Theorems is the exact translation of theorems from [5]. The respective proofs, however, do not change in any way, if we regard quasi ordered structures instead of partially ordered ones as the anti-symmetry of the relation R is not needed (with the exception of the \Leftarrow implication of the part on commutativity, which does not hold in this case). The often quoted version of the "Ends lemma" is therefore the version assuming quasi ordered structures.

Definition 2.5. Two semihypergroups (S, \circ) and (S', \circ') are anti-isomorphic if one is isomorphic to the dual of the other, that is if there exists a bijection $\sigma : S \rightarrow S'$ such that $\sigma(a \circ b) = \sigma(b) \star \sigma(a)$ for all $a, b \in S$; in this case σ is an anti-isomorphism. Let (S, \circ, \leq) and (S', \circ', \leq') be two semihypergroups of the same order. A bijection $\sigma : S \rightarrow S'$ is an order preserving isomorphism if $\sigma(a \circ b) = \sigma(a) \circ' \sigma(b)$ and $a \leq b$ implies that $\sigma(a) \leq' \sigma(b)$ for all $a, b \in S$ and an order preserving anti-isomorphism if $\sigma(a \circ b) = \sigma(b) \circ' \sigma(a)$ and $a \leq b$ implies that $\sigma(a) \leq' \sigma(b)$ for all $a, b \in S$. Two semihypergroups (S, \circ) and (S', \circ') are equivalent if there exists an isomorphism or an anti-isomorphism from one to the other. This is written:

$$(S, \circ) \stackrel{Eq}{\cong} (S', \circ').$$

Often shorter and simpler notations can be used. When the relevant semihypergroup hyper-operations are unambiguous they are omitted and one writes:

$$S \stackrel{Eq}{\cong} S'.$$

Also, Two semihypergroups (S, \circ, \leq) and (S', \circ', \leq') are order preserving equivalent if there exists an order preserving isomorphism or an order preserving anti-isomorphism from one to the other. This is written:

$$(S, \circ, \leq) \stackrel{OPE}{\cong} (S', \circ', \leq').$$

Often shorter and simpler notations can be used. When the relevant semihypergroup hyper-operations are unambiguous they are omitted and one writes:

$$S \stackrel{OPE}{\cong} S'.$$

Corollary 2.6. *Let (S, \circ, \leq) and (S', \circ', \leq') be two semihypergroups and $\sigma : S \rightarrow S'$ is an order preserving isomorphism or an order preserving anti-isomorphism. Then σ^{-1} is an order preserving.*

Proof. Suppose that there exists $c, d \in S'$ such that $c \leq' d$ implies that $\sigma^{-1}(c) \not\leq \sigma^{-1}(d)$. Then $\sigma^{-1}(d) < \sigma^{-1}(d)$ and σ is an order preserving. So $\sigma(\sigma^{-1}(d)) <' \sigma(\sigma^{-1}(d))$. This means that $c \not\leq' d$ and it is a contradiction. \square

Theorem 2.7. *Let (S, \cdot, \leq) and (S', \cdot', \leq') be order preserving equivalent semigroups then (S, \circ) and (S', \circ') are equivalent semihypergroups.*

Proof. Let $\sigma : (S, \cdot, \leq) \rightarrow (S', \cdot', \leq')$ be an isomorphism or anti-isomorphism. We define $\bar{\sigma} : (S, \circ) \rightarrow (S', \circ')$ by $\bar{\sigma}(a) = \sigma(a)$. If σ is an anti-isomorphism then

$$\begin{aligned} \bar{\sigma}(a \circ b) &= \bar{\sigma}(\{x \in S, a \cdot b \leq x\}) \\ &= \{\sigma(x) \in S', a \cdot b \leq x\}, \\ \bar{\sigma}(b) \circ' \bar{\sigma}(a) &= \{y \in S', \sigma(b) \cdot' \sigma(a) \leq' y\} \\ &= \{y \in S', \sigma(a \cdot b) \leq' y\} \\ &= \{\sigma(x) \in S', \sigma(a \cdot b) \leq' \sigma(x)\} \\ &= \{\sigma(x) \in S', (a \cdot b) \leq x\}. \end{aligned}$$

Then $\bar{\sigma}(a \circ b) = \bar{\sigma}(b) \circ' \bar{\sigma}(a)$. \square

3. MAIN RESULTS

In that section, we demonstrate our intention in 4 steps.

- 1) First of all, we count all non-empty quasi order relation on the set $H = \{1, 2, 3\}$ which are 29 and name them R_1, R_2, \dots, R_{29} .
- 2) Chotchaisthit in [3] showed that there are only 24 semigroups of order 3 which we name them S_1, \dots, S_{24} and list them for future use.

- 3) In third step, by a computer MATLAB program software, we examine which of the all $24 \times 29 = 672$ triple (S_i, \cdot, R_j) for all $i \in \{1, 2, \dots, 24\}$ and $j \in \{1, 2, \dots, 29\}$ can becomes a quasi ordered semigroup.
- 4) Finally, we use the Ends lemma construction and another procedure in MATLAB in order to determine all EL -hypergroups of order 3.

3.1. step 1 (quasi ordered relations). Let $H = \{1, 2, 3\}$. Because the Cartesian product $H \times H$ has cardinality 9, there exist $2^9 - 1 = 511$ possible nonempty relations on H . However, only reflexive relations are considered, since quasi-orders must satisfy reflexivity. It is easy to check that, the number of reflexive relations on H is $2^6 = 64$ (the number of subsets of the set $\{(x, y) | x, y \in H, x \neq y\}$). Among these 64 relation, we should find transitive ones. In order to do this, consider the following points:

- 1) Clearly, the relations $R_1 = \{(x, x), x \in H\}$ and $R_2 = H \times H$ are transitive.
- 2) There are 6 reflexive relations of order 8. All of them are not transitive. (Suppose R is a reflexive relation of order 8 and $(x, y) \notin R, (x \neq y)$. Since for any $z \in H$, it holds $(x, z) \in R$ and $(z, y) \in R$ we can see that R is not transitive.)
- 3) Among all 15 reflexive relations of cardinal 7 on H , there are only 6 relations which are transitive. We list them in the following:

$$R_3 = \{(1, 1), (2, 2), (3, 3), (3, 1), (1, 3), (3, 2), (1, 2)\}$$

$$R_4 = \{(1, 1), (2, 2), (3, 3), (3, 1), (1, 3), (2, 3), (2, 1)\}$$

$$R_5 = \{(1, 1), (2, 2), (3, 3), (1, 3), (1, 2), (2, 3), (2, 1)\}$$

$$R_6 = \{(1, 1), (2, 2), (3, 3), (1, 3), (1, 2), (2, 3), (3, 2)\}$$

$$R_7 = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1), (3, 1), (3, 2)\}$$

$$R_8 = \{(1, 1), (2, 2), (3, 3), (3, 1), (2, 1), (3, 2), (2, 3)\}$$

- 4) There are 20 reflexive relations of cardinal 6 on H . Meanwhile, there are only 6 relations which are quasi order as below:

$$R_9 = \{(1, 1), (2, 2), (3, 3), (2, 3), (1, 3), (1, 2)\}$$

$$R_{10} = \{(1, 1), (2, 2), (3, 3), (1, 3), (3, 2), (1, 2)\}$$

$$R_{11} = \{(1, 1), (2, 2), (3, 3), (3, 1), (3, 2), (1, 2)\}$$

$$R_{12} = \{(1, 1), (2, 2), (3, 3), (2, 1), (2, 3), (3, 1)\}$$

$$R_{13} = \{(1, 1), (2, 2), (3, 3), (3, 1), (2, 1), (3, 2)\}$$

$$R_{14} = \{(1, 1), (2, 2), (3, 3), (1, 3), (2, 3), (2, 1)\}$$

- 5) There are 9 quasi order relations of cardinal 5 and 6 quasi order relations of cardinal 4 on H . One can find them in the following:

$$R_{15} = \{(1, 1), (2, 2), (3, 3), (3, 1), (1, 3)\}$$

$$R_{16} = \{(1, 1), (2, 2), (3, 3), (1, 2), (1, 3)\}$$

$$R_{17} = \{(1, 1), (2, 2), (3, 3), (2, 3), (1, 3)\}$$

$$R_{18} = \{(1, 1), (2, 2), (3, 3), (2, 1), (1, 2)\}$$

$$R_{19} = \{(1, 1), (2, 2), (3, 3), (3, 2), (3, 1)\}$$

$$R_{20} = \{(1, 1), (2, 2), (3, 3), (3, 2), (2, 3)\}$$

$$R_{21} = \{(1, 1), (2, 2), (3, 3), (2, 1), (2, 3)\}$$

$$R_{22} = \{(1, 1), (2, 2), (3, 3), (2, 1), (3, 1)\}$$

$$R_{23} = \{(1, 1), (2, 2), (3, 3), (1, 2), (3, 2)\}$$

$$R_{24} = \{(1, 1), (2, 2), (3, 3), (3, 1)\}$$

$$R_{25} = \{(1, 1), (2, 2), (3, 3), (1, 3)\}$$

$$R_{26} = \{(1, 1), (2, 2), (3, 3), (3, 2)\}$$

$$R_{27} = \{(1, 1), (2, 2), (3, 3), (2, 3)\}$$

$$R_{28} = \{(1, 1), (2, 2), (3, 3), (2, 1)\}$$

$$R_{29} = \{(1, 1), (2, 2), (3, 3), (1, 2)\}$$

3.2. Step2. (Semigroups of order 3). Chotchaisthit in [3] used elementary ideas and showed that there are only 24 non-isomorphic semigroups of order 3 (24 non-isomorphic semigroups of order 3). Set $S_i = \{1, 2, 3\}$. Although, they are known by their standard names, we name them as $(S_1, \cdot_1), \dots, (S_{24}, \cdot_{24})$, as below:

\cdot_1	1	2	3	\cdot_2	1	2	3	\cdot_3	1	2	3	\cdot_4	1	2	3
1	1	1	1	1	1	1	3	1	1	2	2	1	1	1	1
2	1	1	1	2	1	1	3	2	2	1	1	2	1	1	1
3	1	1	1	3	3	3	1	3	2	1	1	3	1	1	2
\cdot_5	1	2	3	\cdot_6	1	2	3	\cdot_7	1	2	3	\cdot_8	1	2	3
1	1	2	3	1	1	2	3	1	1	2	3	1	1	2	3
2	2	3	1	2	2	2	2	2	2	2	3	2	2	2	2
3	3	1	2	3	3	2	2	3	3	3	2	3	3	2	1
\cdot_9	1	2	3	\cdot_{10}	1	2	3	\cdot_{11}	1	2	3	\cdot_{12}	1	2	3
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	1	2	3	2	1	2	1	2	1	2	1	2	1	2	2
1	1	1	1	3	1	1	1	3	1	3	1	3	1	2	2
\cdot_{13}	1	2	3	\cdot_{14}	1	2	3	\cdot_{15}	1	2	3	\cdot_{16}	1	2	3
1	1	2	1	1	1	1	1	1	1	1	1	1	1	1	1
2	1	2	1	2	2	2	2	2	1	2	1	2	1	2	2
3	1	2	1	3	1	1	1	3	1	1	3	3	1	2	3
\cdot_{17}	1	2	3	\cdot_{18}	1	2	3	\cdot_{19}	1	2	3	\cdot_{20}	1	2	3
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	1	2	2	2	1	2	3	2	2	2	2	2	2	2	2
3	1	3	3	3	1	2	3	3	1	1	3	3	1	2	3
\cdot_{21}	1	2	3	\cdot_{22}	1	2	3	\cdot_{23}	1	2	3	\cdot_{24}	1	2	3
1	1	1	1	1	1	2	1	1	1	2	1	1	1	2	3
2	2	2	2	2	1	2	1	2	1	2	2	2	1	2	3
3	3	3	3	3	1	2	3	3	1	2	3	3	1	2	3

Remark 3.1. We have:

$$\begin{aligned} S_9 &\stackrel{Eq}{\cong} S_{11}, & S_{13} &\stackrel{Eq}{\cong} S_{14}, & S_{17} &\stackrel{Eq}{\cong} S_{18}, \\ S_{19} &\stackrel{Eq}{\cong} S_{23}, & S_{20} &\stackrel{Eq}{\cong} S_{22}, & S_{21} &\stackrel{Eq}{\cong} S_{24}. \end{aligned}$$

3.3. Step 3. (Quasi ordered semigroups). There are $24 \cdot 29 = 696$ triple (S_i, \cdot_i, R_j) for $1 \leq i \leq 24$ and $1 \leq j \leq 29$. But we look for those that are quasi-ordered semigroups (i.e., those that satisfy the monotonicity condition). Using the given procedure in appendix A,

in MATLAB software, we can see that among all 696 triple (S_i, \cdot_i, R_j) for $1 \leq i \leq 24$ and $1 \leq j \leq 29$ there are 364 ones which satisfies monotone condition. We list them in the following:

Proposition 3.2. *Suppose S_i and R_j as the above for $i \in \{1, 2, \dots, 24\}$ and $j \in \{1, 2, \dots, 29\}$. Then in the table 1, (S_i, \cdot_i, R_j) is a quasi group:*

TABLE 1. Quasi ordered semigroups

(S_i, \cdot_i, R_j)	i	j such that (S_i, \cdot_i, R_j) is a quasi-ordered semigroup
(S_i, \cdot_i, R_j)	1, 21, 24	1, ..., 29
(S_i, \cdot_i, R_j)	2	1, ..., 4, 15, 18, 28, 29
(S_i, \cdot_i, R_j)	3	1, 2, 5, 7, 18, 20, 26, 27
(S_i, \cdot_i, R_j)	4	1, 2, 5, 7, 9, 13, 16, 18, 22, 28, 29
(S_i, \cdot_i, R_j)	5	1, 2
(S_i, \cdot_i, R_j)	6	1, 2, 6, 8, 10, 12, 20, 21, 23, 26, 27
(S_i, \cdot_i, R_j)	7	1, 2, 6, 8, 18, 20, 28, 29
(S_i, \cdot_i, R_j)	8	1, ..., 4, 15, 21, 23
(S_i, \cdot_i, R_j)	9, 11	1, ..., 4, 10, 12, 15, 16, 22, 24, 25
(S_i, \cdot_i, R_j)	10, 13, 14	1, ..., 5, 7, 10, 11, 12, 14, 15, 16, 18, 21, ..., 25, 28, 29
(S_i, \cdot_i, R_j)	12, 16	1, 2, 5, ..., 10, 12, 13, 16, 18, 20, ..., 23, 26, ..., 29
(S_i, \cdot_i, R_j)	15, 19, 22	1, ..., 5, 7, 11, 14, 15, 16, 18, 22, 24, 25, 28, 29
(S_i, \cdot_i, R_j)	17, 18	1, 2, 6, 8, 9, 10, 12, 13, 16, 20, 22, 26, 27
(S_i, \cdot_i, R_j)	20, 23	1, 2, 5, 7, 10, 12, 16, 18, 21, 22, 23, 28, 29

Proof. Every (S_i, \cdot_i, R_j) in Table 1 is a quasi ordered semigroups. The remaining (S_i, \cdot_i, R_j) that do not appear in Table 1 are not quasi ordered semigroups. In the table 2, for $i = 5$ and $j = 2, 3, \dots, 29$, we show by counterexamples listed in the second column that (S_i, \cdot_i, R_j) is not a quasi ordered semigroup. The proofs of the other parts are similar and can also be checked in Appendix A. \square

TABLE 2. (S_5, \cdot_5, R_j) , $j \neq 1, 2$ are not quasi ordered semigroups

(S_i, \cdot_i, R_j)	j	Counter example
(S_5, \cdot_5, R_j)	3, 11	$(3, 1) \in R_j$ but $(3 \cdot 3, 1 \cdot 3) \notin R_j$
(S_5, \cdot_5, R_j)	4, 8, 13, 24	$(3, 1) \in R_j$ but $(3 \cdot 2, 1 \cdot 2) \notin R_j$
(S_5, \cdot_5, R_j)	5	$(1, 3) \in R_j$ but $(1 \cdot 3, 3 \cdot 3) \notin R_5$
(S_5, \cdot_5, R_j)	6, 15, 25	$(1, 3) \in R_j$ but $(1 \cdot 2, 3 \cdot 2) \notin R_j$
(S_5, \cdot_5, R_j)	7, 10, 16, 23, 29	$(1, 2) \in R_j$ but $(1 \cdot 2, 2 \cdot 2) \notin R_j$
(S_5, \cdot_5, R_j)	9	$(2, 3) \in R_j$ but $(2 \cdot 2, 3 \cdot 2) \notin R_j$
(S_5, \cdot_5, R_j)	12, 22	$(2, 1) \in R_j$ but $(2 \cdot 3, 1 \cdot 3) \notin R_j$
(S_5, \cdot_5, R_j)	14	$(2, 1) \in R_j$ but $(2 \cdot 2, 1 \cdot 2) \notin R_j$
(S_5, \cdot_5, R_j)	17, 20	$(2, 3) \in R_j$ but $(2 \cdot 3, 3 \cdot 3) \notin R_j$
(S_5, \cdot_5, R_j)	18, 21, 28	$(2, 1) \in R_j$ but $(2 \cdot 2, 2 \cdot 1) \notin R_j$
(S_5, \cdot_5, R_j)	19, 26	$(3, 2) \in R_j$ but $(3 \cdot 3, 3 \cdot 2) \notin R_j$
(S_5, \cdot_5, R_j)	27	$(2, 3) \in R_j$ but $(2 \cdot 2, 2 \cdot 3) \notin R_j$

3.4. Step 4 (Creating *EL*-hypergroups).

In this section, we utilize the Ends Lemma construction on all quasi-ordered semigroups identified in step 3. According to Lemma 2.1, all of these are semihypergroups, but our focus is specifically on those that qualify as hypergroups. A straightforward computation, following the procedure outlined in Appendix B, reveals that there are exactly 27 *EL*-hypergroups, which are listed below. For this section, let $H = \{1, 2, 3\}$. Additionally, we identify the base quasi ordered semigroups responsible for generating each corresponding *EL*-hypergroups.

1) Hypergroup #1 (Total hypergroup) The hypergroup $(H, *_1)$ in Table 3 can be achieved

TABLE 3. Cayley table of the hyperoperation for the *EL*-hypergroup #1

$*_1$	1	2	3
1	H	H	H
2	H	H	H
3	H	H	H

by: (S_1, \cdot_1, R_3) , (S_1, \cdot_1, R_5) , (S_1, \cdot_1, R_6) , (S_1, \cdot_1, R_9) , (S_1, \cdot_1, R_{10}) , $(S_{10}, \cdot_{10}, R_5)$, $(S_{12}, \cdot_{12}, R_5)$, (S_1, \cdot_1, R_{16}) , (S_2, \cdot_2, R_3) and (S_k, \cdot_k, R_2) for all $k = 1, 2, \dots, 29$.

2) Hypergroup #2

The hypergroup $(H, *_2)$ in Table 4 can be achieved by: (S_4, \cdot_4, R_{16}) .

TABLE 4. Cayley table of the hyperoperation for the *EL*-hypergroup #2

$*_2$	1	2	3
1	H	H	H
2	H	H	H
3	H	H	{2}

3) Hypergroup #3

The hypergroup $(H, *_3)$ in Table 5 can be achieved by (S_9, \cdot_9, R_3) , $(S_{10}, \cdot_{10}, R_3)$, $(S_{10}, \cdot_{10}, R_{10})$,

TABLE 5. Cayley table of the hyperoperation for the *EL*-hypergroup #3

$*_3$	1	2	3
1	H	H	H
2	H	{2}	H
3	H	H	H

$(S_{10}, \cdot_{10}, R_{16})$, $(S_{11}, \cdot_{11}, R_3)$ and $(S_{15}, \cdot_{15}, R_3)$.

4) Hypergroup #4

The hypergroup $(H, *_4)$ in Table 6 can be achieved by (S_7, \cdot_7, R_8) and (S_6, \cdot_6, R_8) .

TABLE 6. Cayley table of the hyperoperation for the *EL*-hypergroup #4

$*_4$	1	2	3
1	{1}	H	H
2	H	H	H
3	H	H	H

5) Hypergroup #5

The hypergroup $(H, *_5)$ in Table 7 can be achieved by (S_4, \cdot_4, R_9) .

TABLE 7. Cayley table of the hyperoperation for the EL -hypergroup #5

$*_2$	1	2	3
1	H	H	H
2	H	H	H
3	H	H	{2,3}

6) Hypergroup #6

TABLE 8. Cayley table of the hyperoperation for the EL -hypergroup #6

$*_6$	1	2	3
1	H	H	H
2	H	{2}	H
3	H	H	{3}

The hypergroup $(H, *_6)$ in Table 8 can be achieved by $(S_{15}, \cdot_{15}, R_{16})$.

7) Hypergroup #7

TABLE 9. Cayley table of the hyperoperation for the EL -hypergroup #7

$*_7$	1	2	3
1	H	H	H
2	H	{2}	H
3	H	{3}	H

The hypergroup $(H, *_7)$ in Table 9 can be achieved by $(S_{11}, \cdot_{11}, R_{16})$.

8) Hypergroup #8

TABLE 10. Cayley table of the hyperoperation for the EL -hypergroup #8

$*_8$	1	2	3
1	H	H	H
2	H	{2}	{2,3}
3	H	H	H

The hypergroup $(H, *_8)$ in Table 10 can be achieved by (S_9, \cdot_9, R_{10}) .

9) Hypergroup #9

The hypergroup $(H, *_9)$ in Table 11 can be achieved by $(S_{11}, \cdot_{11}, R_{10})$.

TABLE 11. Cayley table of the hyperoperation for the EL-hypergroup #9

$*_9$	1	2	3
1	H	H	H
2	H	{2}	H
3	H	{2, 3}	H

TABLE 12. Cayley table of the hyperoperation for the EL-hypergroup #10

$*_{10}$	1	2	3
1	H	H	H
2	H	{2, 3}	{2, 3}
3	H	H	H

10) Hypergroup #10

The hypergroup $(H, *_{10})$ in Table 12 can be achieved by (S_9, \cdot_9, R_6) .

11) Hypergroup #11

TABLE 13. Cayley table of the hyperoperation for the EL-hypergroup #11

$*_{11}$	1	2	3
1	H	H	H
2	H	{2}	{3}
3	H	H	H

The hypergroup $(H, *_{11})$ in Table 13 can be achieved by (S_9, \cdot_9, R_{16}) .

12) Hypergroup #12

TABLE 14. Cayley table of the hyperoperation for the EL-hypergroup #12

$*_{12}$	1	2	3
1	{1}	H	{3}
2	H	H	H
3	H	H	H

The hypergroup $(H, *_{12})$ in Table 14 can be achieved by (S_6, \cdot_6, R_{21}) .

13) Hypergroup #13

The hypergroup $(H, *_{13})$ in Table 15 can be achieved by $(S_{16}, \cdot_{16}, R_{10})$.

14) Hypergroup #14

The hypergroup $(H, *_{14})$ in Table 16 can be achieved by $(S_{18}, \cdot_{18}, R_9)$.

TABLE 15. Cayley table of the hyperoperation for the EL -hypergroup #13

$*_{13}$	1	2	3
1	H	H	H
2	H	{2}	{2}
3	H	{2}	{2, 3}

TABLE 16. Cayley table of the hyperoperation for the EL -hypergroup #14

$*_{14}$	1	2	3
1	H	H	H
2	H	{2, 3}	{3}
3	H	{2, 3}	{3}

15) Hypergroup #15

TABLE 17. Cayley table of the hyperoperation for the EL -hypergroup #15

$*_{15}$	1	2	3
1	{1}	H	{3}
2	H	H	H
3	{3}	H	{1}

The hypergroup $(H, *_1$ in Table 3 can be achieved by (S_8, \cdot_8, R_{21}) .

16) Hypergroup #16

The hypergroup $(H, *_{16})$ in Table 18 can be achieved by (S_6, \cdot_6, R_{12}) .

TABLE 18. Cayley table of the hyperoperation for the EL -hypergroup #16

$*_{16}$	1	2	3
1	{1}	H	{1, 3}
2	H	H	H
3	{1, 3}	H	H

17) Hypergroup #17

The hypergroup $(H, *_{17})$ in Table 19 can be achieved by $(S_{12}, \cdot_{12}, R_6)$, $(S_{12}, \cdot_{12}, R_9)$, $(S_{16}, \cdot_{16}, R_6)$, $(S_{17}, \cdot_{17}, R_6)$ and $(S_{18}, \cdot_{18}, R_6)$.

18) Hypergroup #18

The hypergroup $(H, *_{18})$ in Table 20 can be achieved by (S_8, \cdot_8, R_4) .

19) Hypergroup #19

The hypergroup $(H, *_{19})$ in Table 21 can be achieved by $(S_{16}, \cdot_{16}, R_9)$.

20) Hypergroup #20

The hypergroup $(H, *_{20})$ in Table 22 can be achieved by $(S_{17}, \cdot_{17}, R_{16})$.

TABLE 19. Cayley table of the hyperoperation for the EL-hypergroup #17

* ₁₇	1	2	3
1	H	H	H
2	H	{2, 3}	{2, 3}
3	H	{2, 3}	{2, 3}

TABLE 20. Cayley table of the hyperoperation for the EL-hypergroup #18

* ₁₈	1	2	3
1	{1, 3}	H	{1, 3}
2	H	H	H
3	{1, 3}	H	{1, 3}

TABLE 21. Cayley table of the hyperoperation for the EL-hypergroup #19

* ₁₉	1	2	3
1	H	H	H
2	H	{2, 3}	{2, 3}
3	H	{2, 3}	{3}

TABLE 22. Cayley table of the hyperoperation for the EL-hypergroup #20

* ₂₀	1	2	3
1	H	H	H
2	H	{2}	{2}
3	H	{3}	{3}

21) Hypergroup #21

The hypergroup $(H, *_{21})$ in Table 23 can be achieved by $(S_{18}, \cdot_{18}, R_{16})$.

22) Hypergroup #22

The hypergroup $(H, *_{22})$ in Table 24 can be achieved by $(S_{16}, \cdot_{16}, R_{16})$.

23) Hypergroup #23

The hypergroup $(H, *_{23})$ in Table 25 can be achieved by $(S_{12}, \cdot_{12}, R_{10})$ and $(S_{12}, \cdot_{12}, R_{16})$.

24) Hypergroup #24

The hypergroup $(H, *_{24})$ in Table 26 can be achieved by $(S_{17}, \cdot_{17}, R_9)$.

25) Hypergroup #25

The hypergroup $(H, *_{25})$ in Table 27 can be achieved by $(S_{17}, \cdot_{17}, R_{10})$.

26) Hypergroup #26

The hypergroup $(H, *_{26})$ in Table 28 can be achieved by $(S_{18}, \cdot_{18}, R_{10})$.

TABLE 23. Cayley table of the hyperoperation for the EL -hypergroup #21

*21	1	2	3
1	H	H	H
2	H	{2}	{3}
3	H	{2}	{3}

TABLE 24. Cayley table of the hyperoperation for the EL -hypergroup #22

*22	1	2	3
1	H	H	H
2	H	{2}	{2}
3	H	{2}	{3}

TABLE 25. Cayley table of the hyperoperation for the EL -hypergroup #23

*23	1	2	3
1	H	H	H
2	H	{2}	{2}
3	H	{2}	{2}

TABLE 26. Cayley table of the hyperoperation for the EL -hypergroup #24

*24	1	2	3
1	H	H	H
2	H	{2, 3}	{2, 3}
3	H	{3}	{3}

27) Hypergroup #27

The hypergroup $(H, *_{27})$ in Table 29 can be achieved by (S_3, \cdot_3, R_{20}) and (S_3, \cdot_3, R_{27}) .

28) Hypergroup #28

The hypergroup $(H, *_{28})$ in Table 30 can be achieved by (S_7, \cdot_7, R_{29}) .

29) Hypergroup #29

The hypergroup $(H, *_{29})$ in Table 31 can be achieved by (S_2, \cdot_2, R_{18}) , (S_2, \cdot_2, R_{29}) and (S_7, \cdot_7, R_{18}) .

30) Hypergroup #30 (Table 32)

The hypergroup $(H, *_{30})$ in Table 32 can be achieved by $(S_{15}, \cdot_{15}, R_5)$, $(S_{16}, \cdot_{16}, R_5)$, $(S_{19}, \cdot_{19}, R_5)$, $(S_{20}, \cdot_{20}, R_5)$, $(S_{22}, \cdot_{22}, R_5)$ and $(S_{23}, \cdot_{23}, R_5)$

31) Hypergroup #31 (Group \mathbb{Z}_3) The hypergroup $(H, *_{31})$ in Table 33 can be achieved by (S_5, \cdot_5, R_1) in the scene of EL -construction.

Theorem 3.3. *Three hypergroups $(H, *_{31})$, $(H, *_{32})$ and $(H, *_{33})$ (Tables 5, 6 and 32) are isomorphic.*

Proof. Let $f = (12) \in S_3$. A direct computation shows that

$$f : (H, *_{31}) \longrightarrow (H, *_{32})$$

TABLE 27. Cayley table of the hyperoperation for the EL-hypergroup #25

*25	1	2	3
1	H	H	H
2	H	{2}	{2}
3	H	{2, 3}	{2, 3}

TABLE 28. Cayley table of the hyperoperation for the EL-hypergroup #26

*26	1	2	3
1	H	H	H
2	H	{2}	{2, 3}
3	H	{2}	{2, 3}

TABLE 29. Cayley table of the hyperoperation for the EL-hypergroup #27

*27	1	2	3
1	{1}	{2, 3}	{2, 3}
2	{2, 3}	{1}	{1}
3	{2, 3}	{1}	{1}

TABLE 30. Cayley table of the hyperoperation for the EL-hypergroup #28

*28	1	2	3
1	{1}	{1, 2}	{3}
2	{1, 2}	{1, 2}	{3}
3	{3}	{3}	{1, 2}

is an isomorphism. Likewise, if $f = (23) \in S_3$, then

$$f : (H, *_3) \longrightarrow (H, *_30)$$

is also an isomorphism. □

Theorem 3.4. *Two hypergroups $(H, *_17)$ and $(H, *_18)$ (Tables 19 and 20) are isomorphic.*

Proof. Let $f = (13) \in S_3$. It is straightforward to verify that f is an isomorphism; consequently,

$$(H, *_17) \cong (H, *_18).$$
□

Theorem 3.5. *Two hypergroups $(H, *_20)$ and $(H, *_21)$ (Tables 22 and 23) are the same via isomorphism.*

Proof. Let $f = (23) \in S_3$. Viewing f as an isomorphism

$$f : (H, *_20) \longrightarrow (H, *_21),$$

the verification is straightforward. □

Corollary 3.6. *There are only 27 non-isomorphic EL-hypergroups of order 3.*

Theorem 3.7. *The group \mathbb{Z}_3 is an EL-hypergroup (Table 33) of order 3 which can be constructed by only one way in EL-construction.*

TABLE 31. Cayley table of the hyperoperation for the EL -hypergroup #29

*29	1	2	3
1	{1, 2}	{1, 2}	{3}
2	{1, 2}	{1, 2}	{3}
3	{3}	{3}	{1, 2}

TABLE 32. Cayley table of the hyperoperation for the EL -hypergroup #30

*30	1	2	3
1	H	H	H
2	H	H	H
3	H	H	{3}

TABLE 33. Cayley table of the hyperoperation for the EL -hypergroup #31

*31	1	2	3
1	{1}	{2}	{3}
2	{2}	{3}	{1}
3	{3}	{1}	{2}

Proof. According to 3.2, (S_5, \cdot_5, R_1) and (S_5, \cdot_5, R_2) are the two possible cases for $(\mathbb{Z}_3, +)$ to be a quasi ordered semigroup. If we set EL -construction on (S_5, \cdot_5, R_2) , we get the total hypergroup $(H, *_1)$. Finally, if we set EL -construction on (S_5, \cdot_5, R_1) , the resulted hypergroup would be $(H, *_31)$ which is clearly \mathbb{Z}_3 . \square

Remark 3.8. Assume that each semihypergroup (S_i, \cdot_i, R_j) induces an EL -hyperstructure $(H, *)$. We denote this by

$$EL(S_i, \cdot_i, R_j) = (H, *).$$

Then the following statements hold.

- (1) If $(S_k, \cdot_k, R_2) \stackrel{OPE}{\cong} (S_l, \cdot_l, R_2)$ then

$$EL(S_k, \cdot_k, R_2) \stackrel{Eq}{\cong} EL(S_l, \cdot_l, R_2),$$

when $k, l \in \{1, 2, \dots, 29\}$.

- (2) If $(S_9, \cdot_9, R_k) \stackrel{OPE}{\cong} (S_{11}, \cdot_{11}, R_k)$ then

$$EL(S_9, \cdot_9, R_k) \stackrel{Eq}{\cong} EL(S_{11}, \cdot_{11}, R_k),$$

when $k \in \{3, 10, 17\}$.

- (3) If $(S_{17}, \cdot_{17}, R_k) \stackrel{OPE}{\cong} (S_{18}, \cdot_{18}, R_k)$ then

$$EL(S_{17}, \cdot_{17}, R_k) \stackrel{Eq}{\cong} EL(S_{18}, \cdot_{18}, R_k),$$

when $k \in \{5, 6, 9, 10, 17\}$.

4. CONCLUSIONS

In this paper, we enumerated all possible EL -hypergroups with 3 elements. Specifically, we demonstrated that among the 23,192 hypergroups of order 3, there are only 27 non-isomorphic hypergroups, up to isomorphism, that satisfy the Ends Lemma construction.

In future work, we plan to explore the enumeration of EL -hypergroups with more than 3 elements. Our goal is to identify the relationship between the cardinality of a given set H and the number of EL -hypergroups on H , if such a relationship exists. Additionally, future research ideas include the comprehensive identification of all EL -semihypergroups and EL -hypermonoids, which could be of significant interest to researchers in this field.

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5. APPENDIX

In this section, two appendixes are given. In the first one, a procedure in MATLAB is given by which one can find all possible triples (S_i, \cdot_i, R_j) which are quasi ordered semigroup. By the second one, the EL -hyperstructure $(H, *_i)$, associated to (S_i, \cdot_i, R_j) , can be achieved.

5.1. Appendix A (quasi ordered semigroup test). .

```
%% ===== Initialization =====
clear all; clc;

%% ===== Relations =====
R1=[1 1;2 2;3 3];
R2=[1 1;2 2;3 3;3 1;3 2;2 1;2 3;1 2;1 3];
R3=[1 1;2 2;3 3;3 1;3 2;1 3;1 2];
R4=[1 1;2 2;3 3;3 1;1 3;2 3;2 1];
R5=[1 1;2 2;3 3;1 3;1 2;2 3;2 1];
R6=[1 1;2 2;3 3;1 3;1 2;2 3;3 2];
R7=[1 1;2 2;3 3;1 2;2 1;3 1;3 2];
R8=[1 1;2 2;3 3;3 1;2 1;3 2;2 3];
R9=[1 1;2 2;3 3;2 3;1 3;1 2];
R10=[1 1;2 2;3 3;3 2;1 3;1 2];
R11=[1 1;2 2;3 3;3 2;3 1;1 2];
R12=[1 1;2 2;3 3;2 1;2 3;3 1];
R13=[1 1;2 2;3 3;3 1;2 1;3 2];
R14=[1 1;2 2;3 3;2 1;1 3;2 3];
R15=[1 1;2 2;3 3;3 1;1 3];
R16=[1 1;2 2;3 3;1 2;1 3];
R17=[1 1;2 2;3 3;2 3;1 3];
R18=[1 1;2 2;3 3;2 1;1 2];
R19=[1 1;2 2;3 3;3 2;3 1];
R20=[1 1;2 2;3 3;3 2;2 3];
R21=[1 1;2 2;3 3;2 1;2 3];
R22=[1 1;2 2;3 3;2 1;3 1];
R23=[1 1;2 2;3 3;1 2;3 2];
R24=[1 1;2 2;3 3;3 1];
```

```

R25=[1 1;2 2;3 3;1 3];
R26=[1 1;2 2;3 3;3 2];
R27=[1 1;2 2;3 3;2 3];
R28=[1 1;2 2;3 3;2 1];
R29=[1 1;2 2;3 3;1 2];

```

```

%% ===== Semigroups =====

```

```

S1=[1 1 1;1 1 1;1 1 1];
S2=[1 1 3;1 1 3;3 3 1];
S3=[1 2 2;2 1 1;2 1 1];
S4=[1 1 1;1 1 1;1 1 2];
S5=[1 2 3;2 3 1;3 2 1];
S6=[1 2 3;2 2 2;3 2 2];
S7=[1 2 3;2 2 3;3 3 2];
S8=[1 2 3;2 2 2;3 2 1];
S9=[1 1 1;1 2 3;1 1 1];
S10=[1 1 1;1 2 1;1 1 1];
S11=[1 1 1;1 2 1;1 3 1];
S12=[1 1 1;1 2 2;1 2 2];
S13=[1 2 1;1 2 1;1 2 1];
S14=[1 1 1;2 2 2;1 1 1];
S15=[1 1 1;1 2 1;1 1 3];
S16=[1 1 1;1 2 2;1 2 3];
S17=[1 1 1;1 2 2;1 3 3];
S18=[1 1 1;1 2 3;1 2 3];
S19=[1 1 1;2 2 2;1 1 3];
S20=[1 1 1;2 2 2;1 2 3];
S21=[1 1 1;2 2 2;3 3 3];
S22=[1 2 1;1 2 1;1 2 3];
S23=[1 2 1;1 2 2;1 2 3];
S24=[1 2 3;1 2 3;1 2 3];

```

```

%% ===== Interactive Input =====

```

```

m = input('Enter the semigroup number (1-24): ');

```

```

S = Semigroups{m};

```

```

fprintf('\nChecking which relations make S%d a quasi ordered semigroup...\n\n', m);

```

```

cnt = 0;

```

```

for l = 1:length(Relations)

```

```

    R = Relations{l};

```

```

nR = size(R,1);

cntRP = zeros(3,nR);
cntLP = zeros(3,nR);

for i = 1:3
    for j = 1:nR
        RP = [S(R(j,1),i), S(R(j,2),i)];
        LP = [S(i,R(j,1)), S(i,R(j,2))];

        for k = 1:nR
            if isequal(RP,R(k,:))
                cntRP(i,j) = 1;
            end
            if isequal(LP,R(k,:))
                cntLP(i,j) = 1;
            end
        end
    end
end

if all(cntRP(:)==1) && all(cntLP(:)==1)
    cnt = cnt + 1;
    fprintf('(Relation R%d produces a quasi ordered semigroup with S%d\n', 1, m);
end

end

if cnt==0
    fprintf('No relation produces a quasi ordered semigroup with S%d\n', m);
else
    fprintf('\nTotal quasi ordered semigroups found for S%d: %d\n', m, cnt);
end

```

5.2. Appendix B (Generating hypergroups). .

By the following procedure in MATLAB, one can visualize all EL -hyperstructures $(H, *_k)$ associated with (S_i, \cdot, R_j) and easily determine which ones are hypergroups. In addition, this procedure can alert user when the entered triple (S_i, \cdot, R_j) is not a quasi ordered semigroup.

```

%% ===== Full Quasi-Ordered Semigroup + EL-product =====
clear all; clc;

%% ===== Relations R1-R29 =====
R1=[1 1;2 2;3 3];
R2=[1 1;2 2;3 3;3 1;3 2;2 1;2 3;1 2;1 3];
R3=[1 1;2 2;3 3;3 1;3 2;1 3;1 2];

```

```

R4=[1 1;2 2;3 3;3 1;1 3;2 3;2 1];
R5=[1 1;2 2;3 3;1 3;1 2;2 3;2 1];
R6=[1 1;2 2;3 3;1 3;1 2;2 3;3 2];
R7=[1 1;2 2;3 3;1 2;2 1;3 1;3 2];
R8=[1 1;2 2;3 3;3 1;2 1;3 2;2 3];
R9=[1 1;2 2;3 3;2 3;1 3;1 2];
R10=[1 1;2 2;3 3;3 2;1 3;1 2];
R11=[1 1;2 2;3 3;3 2;3 1;1 2];
R12=[1 1;2 2;3 3;2 1;2 3;3 1];
R13=[1 1;2 2;3 3;3 1;2 1;3 2];
R14=[1 1;2 2;3 3;2 1;1 3;2 3];
R15=[1 1;2 2;3 3;3 1;1 3];
R16=[1 1;2 2;3 3;1 2;1 3];
R17=[1 1;2 2;3 3;2 3;1 3];
R18=[1 1;2 2;3 3;2 1;1 2];
R19=[1 1;2 2;3 3;3 2;3 1];
R20=[1 1;2 2;3 3;3 2;2 3];
R21=[1 1;2 2;3 3;2 1;2 3];
R22=[1 1;2 2;3 3;2 1;3 1];
R23=[1 1;2 2;3 3;1 2;3 2];
R24=[1 1;2 2;3 3;3 1];
R25=[1 1;2 2;3 3;1 3];
R26=[1 1;2 2;3 3;3 2];
R27=[1 1;2 2;3 3;2 3];
R28=[1 1;2 2;3 3;2 1];
R29=[1 1;2 2;3 3;1 2];

```

```

%% ===== Semigroups S1-S24 =====

```

```

S1=[1 1 1;1 1 1;1 1 1];
S2=[1 1 3;1 1 3;3 3 1];
S3=[1 2 2;2 1 1;2 1 1];
S4=[1 1 1;1 1 1;1 1 2];
S5=[1 2 3;2 3 1;3 2 1];
S6=[1 2 3;2 2 2;3 2 2];
S7=[1 2 3;2 2 3;3 3 2];
S8=[1 2 3;2 2 2;3 2 1];
S9=[1 1 1;1 2 3;1 1 1];
S10=[1 1 1;1 2 1;1 1 1];
S11=[1 1 1;1 2 1;1 3 1];
S12=[1 1 1;1 2 2;1 2 2];
S13=[1 2 1;1 2 1;1 2 1];
S14=[1 1 1;2 2 2;1 1 1];
S15=[1 1 1;1 2 1;1 1 3];

```

```

S16=[1 1 1;1 2 2;1 2 3];
S17=[1 1 1;1 2 2;1 3 3];
S18=[1 1 1;1 2 3;1 2 3];
S19=[1 1 1;2 2 2;1 1 3];
S20=[1 1 1;2 2 2;1 2 3];
S21=[1 1 1;2 2 2;3 3 3];
S22=[1 2 1;1 2 1;1 2 3];
S23=[1 2 1;1 2 2;1 2 3];
S24=[1 2 3;1 2 3;1 2 3];

%% ===== Interactive Input =====
m = input('Enter semigroup number (1-24): ');
l = input('Enter relation number (1-29): ');

S = Semigroups{m};
R = Relations{l};

%% ===== Check Quasi-Ordered Semigroup =====
cntRP = zeros(3,size(R,1));
cntLP = zeros(3,size(R,1));

for i = 1:3
    for j = 1:size(R,1)
        RP = [S(R(j,1),i), S(R(j,2),i)];
        LP = [S(i,R(j,1)), S(i,R(j,2))];
        for k = 1:size(R,1)
            if isequal(RP,R(k,:))
                cntRP(i,j)=1;
            end
            if isequal(LP,R(k,:))
                cntLP(i,j)=1;
            end
        end
    end
end

if any(cntRP(:)==0) || any(cntLP(:)==0)
    error(['The selected semigroup S',num2str(m),...
        ' with relation R',num2str(l),' cannot form a quasi-ordered semigroup.']);
end

%% ===== EL-product Calculation =====

```

```

ELproduct = cell(3,3);
for i = 1:3
    for j = 1:3
        idx = find(R(:,1)==S(i,j));
        ELproduct{i,j} = R(idx,2);
    end
end

%% ===== Display EL-product =====
disp(['EL-product for S', num2str(m), ' and R', num2str(1), ':']);
for i = 1:3
    row = '';
    for j = 1:3
        row = [row, '{', num2str(ELproduct{i,j}), '}, ']; %#ok<AGROW>
    end
    disp(row)
end

```

REFERENCES

- [1] G. Brinkmann and B. D. McKay, Counting Unlabelled Topologies and Transitive Relations, *Journal of Integer Sequences*, **8** (2005), no. 2, Article 05.2.1.
- [2] A. H. Clifford and G. B. Preston, *The Algebraic Theory of Semigroups. Volume I*, Mathematical Surveys and Monographs, Vol. 7, American Mathematical Society, 1961.
- [3] S. Chotchaisthit, Simple proofs determining all nonisomorphic semigroups of order 3, *App. Math. Sci.* Vol. 8(26) (2014), 1261-1269.
- [4] J. Chvalina, Commutative hypergroups in the sense of Marty and ordered Sets, *Gen. Alg. and Ordered Sets, Proc. Inter. Conf., Olomouc*, (1994).
- [5] J. Chvalina, *Functional Graphs, Quasi-Ordered Sets and Commutative Hypergroups*, Masaryk University, Brno, 1995 (in Czech).
- [6] P. Conrad, Ordered semigroups, *Nagoya Mathematical Journal*, **16** (1960), 51–64.
- [7] P. Corsini and V. Leoreanu, *Applications of Hyperstructure Theory*, Advances in Mathematics, Kluwer Academic Publishers, Dordrecht, 2003.
- [8] P. Corsini, Hyperstructures associated with ordered sets, *Bull. Greek Math. Soc.*, **48** (2003), 7-18.
- [9] B. Davvaz, *Polygroup Theory and Related System*, World Scientific Publishing Co. Pte. Ltd., 2012.
- [10] A. Distler, A. D. Jefferson, M. Kelsey, and L. Kotthoff, The Semigroups of Order 10, in *Principles and Practice of Constraint Programming (CP 2012)*, Lecture Notes in Computer Science, Vol. 7514, Springer, 2012.
- [11] S. K. Das, A Machine Representation of Finite T_0 Topologies, *Journal of the ACM*, **24** (1977), no. 4, 676–692.
- [12] M. Ern e, Struktur- und Anzahlformeln f ur Topologien auf endlichen Mengen, *manuscripta mathematica*, **11** (1974), 221–259.
- [13] M. Ern e and K. Stege, Counting Finite Posets and Topologies, *Order*, **8** (1991), 247–265.
- [14] J. W. Evans, F. Harary, and M. S. Lynn, On the Computer Enumeration of Finite Topologies, *Communications of the ACM*, **10** (1967), no. 5, 295–298.
- [15] G. E. Forsythe, SWAC computes 126 distinct semigroups of order 4, *Proc. Amer. Math. Soc.*, **6** (1955), 443–447.

- [16] S. H. Ghazavi, S. M. Anvarieh and S. Mirvakili, EL^2 -hyperstructures derived from (partially) quasi ordered hyperstructures, *Iranain Journal of Mathematical Sciences and Informatics*, **10** (2015), no. 2, 99-114.
- [17] S. H. Ghazavi, S. M. Anvarieh and S. Mirvakili, Ideals in EL -semihypergroups associated to ordered semigroups, *Journal of Algebraic Systems*, **3** (2016), no. 2, 109–125.
- [18] S. H. Ghazavi and S. M. Anvarieh, EL -hyperstructures associated to n -ary relations, *Soft Computing*, **21** (2013), 5841-5850.
- [19] S. H. Ghazavi and S. Mirvakili, On enumeration of EL -hyperstructures with 2 elements, *Journal of Mahani Mathematical Research Center*, (2022), 11(1), 145 - 158.
- [20] S. H. Ghazavi and S. Mirvakili, On enumeration and classification of EL^2 -semihypergroups and EL^2 - H_v -semihypergroups with 2 elements, *Journal of Algebra and related topics*, (2023), 11(2), 85 - 97.
- [21] S. H. Ghazavi and S. Mirvakili, Ends Lemma based Γ -semihypergroups and their various ideals, *Journal of Algebraic Hyperstructures and Logical Algebras*, (2025), Vol. 6, no. 2, 139-149.
- [22] C. Hollings, *Mathematics across the Iron Curtain: A History of the Algebraic Theory of Semigroups*, History of Mathematics, Vol. 41, American Mathematical Society, 2014.
- [23] D. Heidari and B. Davvaz, On ordered hyperstructures, *U. P. B. Sci. Bull. Series A*, **73** (2011), no. 2.
- [24] S. Hoskova, Binary Hyperstructures Determined by Relational and Transformation Systems, Diss, Faculty of Science, University of Ostrava, 2008.
- [25] D. J. Kleitman, B. R. Rothschild, and J. H. Spencer, The Number of Semigroups of Order n , *Proceedings of the American Mathematical Society*, **55** (1976), no. 1, 227–232.
- [26] V. Krishnamurthy, On the Number of Topologies on a Finite Set, *The American Mathematical Monthly*, **73** (1966), no. 2, 154–157.
- [27] D. Kleitman and B. Rothschild, The Number of Finite Topologies, *Proceedings of the American Mathematical Society*, **25** (1970), no. 2, 276–282.
- [28] D. J. Kleitman and B. L. Rothschild, Asymptotic Enumeration of Partial Orders on a Finite Set, *Transactions of the American Mathematical Society*, **205** (1975), 205–220.
- [29] C. Massouros and G. Massouros, On the borderline of fields and hyperfields, *Mathematics*, **11** (2023), 1289.
- [30] C. Massouros and G. Massouros, On the borderline of fields and hyperfields, part II—enumeration and classification of the hyperfields of order 7, *AIMS Mathematics*, **10** (2025), 21287–21421.
- [31] R. Migliorato, *Ipergruppi di cardinalit 3 e isomorfismi di ipergruppidi commutativi totalmente regolari*, Atti Convegno su Ipergruppi, Udine 1985.
- [32] H. Mirabdollahi, S. M. Anvariye and S. Mirvakili, The associated hyperringoid to a Krasner hyperring, *Journal of Taibah University for Science*, 12:3, 348-356.
- [33] W. Nakkhasen and B. Pibaljomme Hyperideals in EL -semihyperrings, *Thai Journal of Mathematics*, (2017), 133-143.
- [34] G. Nordo, An algorithm on number of isomorphism classes of hypergroups of order 3, *Italian J. Pure Appl. Math.* 2 (1997) 3742.
- [35] M. Novak, Potential of the Ends lemma to create ring-like hyperstructures from quasi-ordered semigroups, *South Bohemia Mathem. Letters*, **17** (2009), no. 1, 39-50.
- [36] M. Novak, The notion of subhyperstructure of "Ends lemma"-based hyperstructures, *Journal of Applied Mathematics*, **3** (2010), no. 2, 237-247.
- [37] M. Novak, Important elements of EL -hyperstructures, in *APLIMAT:10th International Conference*, STU in Bratislava, Bratislava, (2011), 151-158.
- [38] M. Novak, EL -hyperstructures: an overview, *Ratio Mathematica*, **23** (2012), 65-80.
- [39] M. Novak, Some basic properties of EL -hyperstructure, *European Journal of Combinatorics*, **34** (2013), 446-459.
- [40] G. Pfeiffer, Counting Transitive Relations, *Journal of Integer Sequences*, **7** (2004), no. 3, Article 04.3.2.
- [41] M. Novak, On EL -semihypergroup, *European Journal of Combinatorics*, **44** (2015), 274-286.
- [42] P. Rackova, Hypergroups of symmetric matrices, *10th International Congress of Algebraic Hyperstructures and Applications*, Proceeding of AHA (2008).

- [43] I. G. Rosenberg, Hypergroups and join spaces determined by relations, *Italian Journal of Pure and Applied Mathematics*, **4** (1998), 93-101.
- [44] H. Sharp, Jr., Quasi-Orderings and Topologies on Finite Sets, *Proceedings of the American Mathematical Society*, **17** (1966), no. 6, 1344–1349.
- [45] R. E. Stong, Finite Topological Spaces, *Transactions of the American Mathematical Society*, **123** (1966), no. 2, 325–340.
- [46] Ch. Tsitouras and Ch. G. Massouros, On enumeration of hypergroups of order 3, *Comput. Math. Appl.* 59 (2010) 519-523.
- [47] T. Vougiouklis, Representation of hypergroups by generalized permutations, *Algebra Universalis*, **29** (1992), 172-183.