DOI: 10.22098/jhs.2026.17186.1086

Journal of Hyperstructures, Article in Press

Journal Homepage: https://jhs.uma.ac.ir/

Research Paper

SOME RESULTS IN PRODUCT MV-ALGEBRAS

FERESHTEH FOROUZESH[1,*] AND SOMAYEH HADJIREZAEI[2]

[1]Department of Pure Mathematics and Calculations, Faculty of Mathematics, Higher Education Complex of Bam, Kerman, Iran, frouzesh@bam.ac.ir

[2]Department of Mathematics, Vali-e-Asr University of Rafsanjan, P.O. Box 7718897111, Rafsanjan, Iran, s.hajirezaei@vru.ac.ir

ARTICLE INFO

Article history:
Received: 14 April 2025
Accepted: 23 February 2026
Communicated by Yuming Feng

Keywords:
Product MV-algebra
$\cdot$-ideal
$\cdot$-closed system

MSC:
03B50, 06D35.

ABSTRACT

In this paper, we study $\cdot$-ideals in product MV-algebras. We define a $\cdot$-closed system. By the notion of $\cdot$-closed systems, we show that there exists a congruence relation on product MV-algebras. We prove that if P is an $\cdot$-prime ideal of A such that $S = A \setminus P$, then $A[S]$ is a local product MV-algebra (fraction relative to a $\cdot$-closed systems).

1. Introduction

In 1958, Chang defined MV-algebras and in 1959 he also proved the completeness theorem which states the real unit interval $[0, 1]$ as a standard model of this logic [2]. The structures directly obtained from Łukasiewicz logic, in the sense that basic operations coincide with the basic logical connectives (implication and negation), were defined by Font, Rodriguez and Torrens in [8] under the name of Wajsberg algebras.

A. Dvurečenskij and A. Di Nola in [4] introduced the notion of product MV-algebras, that is MV-algebras whose product operation $(\cdot)$ is defined on the whole MV-algebra. This

$*$Address correspondence to F. Forouzesh; Department of Pure Mathematics and Calculations, Faculty of Mathematics, Higher Education Complex of Bam, Kerman, Iran, E-mail: frouzesh@bam.ac.ir.

operation is associative and left/right distributive with respect to partially defined addition. They showed that the category of product MV-algebras is categorically equivalent to the category of associative unital ℓ-rings. In addition, they introduced and studied MVF-algebras [4]. They also introduced $\cdot$-ideals in product MV-algebras. Then they showed that: Any MVF-algebra is a subdirect product of subdirectly irreducible MVF-algebras [4, Corollary 5.6]. Thus they concluded that a product MV-algebra is an MVF-ring if and only if it is a subdirect product of linearly ordered product MV-algebras [4, Theorem 5.8]. S. Saidi Goraghani and R. A. Borzooei studied injective MV -modules and they investigated some conditions for constructing injective MV - modules [14]. Also, H. banivaheb and A. borumand Saeid proved that every $\cdot$-prime ideal in a PMV-subalgebra is the intersection of a $\cdot$-prime ideal in the overalgebra with the subalgebra and they showed that if P is a $\cdot$-prime ideal of a unital PMV-algebra A and A' is a subalgebra of A, having P as a maximal $\cdot$-ideal, then $A'/0_P(A)$ is a local PMV-algebra which is called the localization of A at P relative to A', where $0_P(A)$ is the intersection of all $\cdot$-prime ideals of A contained in P.
In the present paper, we investigate $\cdot$-ideals in product MV-algebras, we study the $\cdot$-ideals of a product MV-algebra A generated by a subset of A and we introduce the notion of $\cdot$-closed system of a product MV-algebra A. Moreover, we show that if P is an $\cdot$-prime ideal of A and $S = A \setminus P$, then $A[S]$ is a local product MV-algebra and $A_P = A[S]$ is a localization at P. Hence we introduce the notion of product MV-algebra of fractions relative to a $\cdot$-closed system.

2. Preliminaries

In this section, we summarize the basic concepts on MV-algebras and product MV-algebras. For more details on these concepts, we refer the reader to [2], [3]-[7] and [13].

Definition 2.1. [2] An MV-algebra is a structure $(M, \oplus, ^*, 0)$ where $\oplus$ is a binary operation, *, is a unary operation and 0 is a constant such that the following conditions are satisfied for any $a, b \in M$:
$(MV1)$ $(M, \oplus, 0)$ is a commutative monoid,
$(MV2)$ $(a^*)^* = a$,
$(MV3)$ $0^* \oplus a = 0^*$,
$(MV4)$ $(a^* \oplus b)^* \oplus b = (b^* \oplus a)^* \oplus a$.

Define the constant $1 = 0^*$ and the auxiliary operations $\odot, \vee$ and $\wedge$ by:

$$a \odot b = (a^* \oplus b^*)^*, \quad a \vee b = a \oplus (b \odot a^*), \quad a \wedge b = a \odot (b \oplus a^*), \quad (a^*)^* = a^{**}.$$

It is shown that $(M, \odot, 1)$ is an abelian monoid and the structure $(M, \vee, \wedge, 0, 1)$ is a bounded distributive lattice [13].

In an MV-algebra M, the Chang's distance function is defined as
$d : M \times M \longrightarrow M, \quad d(a, b) := (a \odot b^*) \oplus (b \odot a^*)$.
Note. An element $a \in A$ is called complemented if there is an element $b \in A$ such that $a \vee b = 1$ and $a \wedge b = 0$. We denote the set of complemented of A by $B(A)$.

Lemma 2.2. [13] *Let M be an MV-algebra. If $x, y, z, t \in M$ and d is a Chang's distance function, then*

(1) $x \leq y$ *iff* $y^* \leq x^*$*,*
(2) If $x \leq y$*, then* $x \oplus z \leq y \oplus z$ *and* $x \odot z \leq y \odot z$*,*
(3) $(x \vee y)^* = x^* \wedge y^*$*,* $(x \wedge y)^* = x^* \vee y^*$*,*
(4) $d(x, y) = 0$ *iff* $x = y$*,*
(5) $d(x, 0) = x, d(x, 1) = x^*$*,*
(6) $d(x, z) \leq d(x, y) \oplus d(y, z)$*,*
(7) $d(x^*, y^*) = d(x, y)$*,*
(8) If $x \leq y$ *and* $z \leq t$*, then* $x \oplus z \leq y \oplus t$*,*
(9) If $x \in B(A)$*, then* $x \odot x = x$*,* $x \oplus x = x$ *and* $x \wedge y = x \odot y$*,*
(10) $x \leq y^* \oplus z$ *if and only if* $x \odot y \leq z$*.*

Lemma 2.3. [3] *Let* M *be an* MV*-algebra. For* $x, y \in M$*, the following conditions are equivalent:*
(1) $x^* \oplus y = 1$*,*
(2) $x \odot y^* = 0$*,*
(3) There is an element $z \in M$ *such that* $x \oplus z = y$*,*
(4) $y = x \oplus (y \ominus x)$*, where* $y \ominus x = y \odot x^*$*.*
For any two elements $x, y \in M$*,* $x \leq y$ *iff* x *and* y *satisfy the equivalent conditions (1)-(4) in the above lemma.*

Definition 2.4. [2] An ideal of an MV-algebra M is a nonempty subset I of M satisfying the following conditions:
$(I1)$ If $x \in I$, $y \in M$ and $y \leq x$ then $y \in I$,
$(I2)$ If $x, y \in I$, then $x \oplus y \in I$.
We denote by $Id(M)$ the set of ideals of an MV-algebra M.

Definition 2.5. [3] Let A and B be two MV-algebras. A function $h : A \to B$ is a morphism of MV-algebras if and only if it satisfies the following conditions, for every $a, b \in A$:
(i) $h(0) = 0$,
(ii) $h(a \oplus b) = h(a) \oplus h(b)$,
(iii) $h(a^*) = h(a)^*$.

Definition 2.6. A proper ideal I of an MV-algebra M is called:
• [3] A *prime* ideal if for all $x, y \in M$, $x \wedge y \in I$ implies $x \in I$ or $y \in I$. Equivalently, P is *prime* if and only if for all $x, y \in M$, $x \odot y^* \in I$ or $y \odot x^* \in I$. We denote by $Spec(M)$ the set of all *prime* ideals of M.
• [11] An *obstinate* ideal of M if $x, y \notin I$ imply $x \odot y^* \in I$ and $y \odot x^* \in I$, for all $x, y \in M$.

Lemma 2.7. [11] *Let* I *be a proper ideal of* A*. Then* I *is an obstinate ideal if and only if* $x \in I$ *or* $x^* \in I$*, for all* $x \in A$*.*

Definition 2.8. [3] Let $X \subseteq A$. The ideal of A generated by X will be denoted by $(X]$, that is the smallest ideal of A which contain X.
$I \in Id(A)$ is called a finitely generated ideal, if $I = (x_1, \dots, x_n]$, for some $x_1, \dots, x_n \in A$ and $n \in \mathbb{N}$.

In particular $(a] = \{x \in A : x \leq na, \text{ for some } n \in \mathbb{N}\}$.

For $I \in Id(A)$ and $a \in A - I$, define $I(a) = (I \cup \{a\}]$.

Proposition 2.9. [3, 13] *Let A be an MV-algebra, $x, y \in A$ and $I, J, I_1, \ldots, I_m$ be ideals of A. Then the following statements hold:*
(1) *If $x \leq y$, then $(x] \subseteq (y]$.*
(2) *$(x] \cap (y] = (x \wedge y]$.*
(3) *$(x] \vee (y] = (x \oplus y]$.*
(4) *If $I_1, \ldots, I_m$ are finitely generated ideals of A, then $I_1 \cap \ldots \cap I_m$ is a finitely generated ideal of A.*
(5) *If $J \subseteq I$ such that I/J and J are finitely generated ideals of A, then I is a finitely generated ideal of A.*
(6) *$I(a) \cap I(b) = I(a \wedge b)$.*

Definition 2.10. [4] A product MV-algebra (or PMV-algebra briefly) is a structure $(A, \oplus, ^*, \cdot, 0)$, where $(A, \oplus, ^*, 0)$ is an MV-algebra and $\cdot$ is a binary associative operation on A such that the following property is satisfied:
if $x + y$ is defined, then $x \cdot z + y \cdot z$ and $z \cdot x + z \cdot y$ are defined and

$$(x + y) \cdot z = x \cdot z + y \cdot z, \quad z \cdot (x + y) = z \cdot x + z \cdot y,$$

where, $+$ is a partial addition on an MV-algebra A as follows: for any $x, y \in M$, $x + y$ is defined if and only if $x \leq y^*$ and in this case,

$$x + y := x \oplus y.$$

The partial addition was defined in [6].

If A is product MV-algebra, then a unity for the product is an element $e \in A$ such that $e \cdot x = x \cdot e = x$ for any $x \in A$. A product MV-algebra that has unity for the product will be called unital.

Lemma 2.11. [6] *For any $x, y, z \in A$ the following properties hold:*
(*a*) *$x + 0 = x$,*
(*b*) *$x \vee y = x + (x^* \odot y)$,*
(*c*) *If $z + x \leq z + y$, then $x \leq y$,*
(*d*) *If $z + x = z + y$, then $x = y$.*

Definition 2.12. [4] An $\cdot$-ideal of product MV-algebra A is an ideal I of MV-algebra A such that if $a \in I$ and $b \in A$, then $a \cdot b \in I$ and $b \cdot a \in I$.
We denote by $Id_p(A)$ the set of $\cdot$-ideals of a product MV-algebra A. The set of all maximal ideals of an MV-algebra A is denoted by $Max(A)$.

Definition 2.13. [9] Let P be an $\cdot$-ideal of of a product MV-algebra A. P is called an $\cdot$-prime ideal, if (i) $P \neq A$ and (ii) for every $a, b \in A$, if $a \cdot b \in P$, then $a \in P$ or $b \in P$.

Lemma 2.14. [4] *If A is a unital product MV-algebra, then:*
(a) The unity for the product is $e = 1$,
(b) $x \cdot y \leq x \wedge y$ for any $x, y \in A$.

Theorem 2.15. [4] *A finite PMV-algebra A admits a product $\cdot$ such that $a \cdot 1 = a = 1 \cdot a$ for any $a \in A$ if and only if A is a Boolean algebra, i.e., $a \oplus a = a$ for any $a \in A$. If it is the case, then $a \cdot b = a \wedge b \in A$.*

Lemma 2.16. [4] *If A is product MV-algebra, then for any $a, b \in A$,*
(i) $a \cdot 0 = 0 = 0 \cdot a$,
(ii) *if $a \leq b$, then for any $c \in A$, $a \cdot c \leq b \cdot c$ and $c \cdot a \leq c \cdot b$.*

3. SOME RESULTS ON $\cdot$-IDEALS IN PRODUCT MV-ALGEBRAS

In the sequel A will be a PMV-algebra, briefly.
In this section, we prove important lemma, that is used in the other theorems and show the general form of the ideal generated by a subset of A. Also, we state isomorphism theorems in PMV-algebras.

Remark 3.1. In general, the union of any family of $\cdot$-ideals of A need not be an $\cdot$-ideal of A.

In the following lemma, we state and prove some properties of PMV-algebras.

Lemma 3.2. *The following properties hold for any $x, y, \alpha \in A$,*
(a) $(nx) \cdot y = x \cdot (ny)$, *for any $n \in \mathbb{N}$, where $nx = x + \cdots + x$, (n-times).*
(b) $x \cdot y^* \leq (x \cdot y)^*$,
(c) $(x \cdot y)^* = x^* \cdot y + (1 \cdot y)^*$,
(d) $(\alpha \cdot x) \odot (\alpha \cdot y)^* \leq \alpha \cdot (x \odot y^*)$,
(e) $\alpha \cdot (x \oplus y) \leq \alpha \cdot x \oplus \alpha \cdot y$,
(f) $d(\alpha \cdot x, \alpha \cdot y) \leq \alpha \cdot d(x, y)$,
Moreover, if A is a unital PMV-algebra, then
$(x \cdot y)^* = x^* \cdot y + y^*$.

Proof. (a) By definition of PMV-algebras, it is clear.
(b) Since $x \cdot 1 \leq 1$, it follows that

$$x \cdot y^* + x \cdot y = x \cdot (y^* + y) = x \cdot 1 \leq 1 = (x \cdot y)^* + x \cdot y,$$

since $+$ is a cancellative, $x \cdot y^* \leq (x \cdot y)^*$.
(c) We have $(x \cdot y)^* \odot (x^* \cdot y)^* = ((x \cdot y) \oplus (x^* \cdot y))^*$. Now, we show that $((x \cdot y) \oplus (x^* \cdot y))^* = ((x \cdot y) + (x^* \cdot y))^*$. By definition of PMV-algebras, we have $(x \cdot y) + (x^* \cdot y) = (x + x^*) \cdot y = 1 \cdot y \leq 1 = (x^* \cdot y) + (x^* \cdot y)^*$, since $+$ is cancellative, we get $(x \cdot y) \leq (x^* \cdot y)^*$. Hence $(x \cdot y) + (x^* \cdot y)$ is defined and in this case, we obtain $(x \cdot y)^* \odot (x^* \cdot y)^* = ((x \cdot y) + (x^* \cdot y))^* = ((x + x^*) \cdot y)^* = (1 \cdot y)^*$. Using (b) it follows that $(x \cdot y)^* = (x \cdot y)^* \vee x^* \cdot y = ((x \cdot y)^* \odot (x^* \cdot y)^*) + x^* \cdot y = (1 \cdot y)^* + x^* \cdot y$.
(d) Since $x, y \leq x \vee y$, by Lemma 2.16 (ii) we get $\alpha \cdot x, \alpha \cdot y \leq \alpha \cdot (x \vee y)$. Hence $\alpha \cdot x \vee \alpha \cdot y \leq \alpha \cdot (x \vee y)$.
Since $(\alpha \cdot x) \odot (\alpha \cdot y)^* \leq (\alpha \cdot y)^*$, by definition of partial addition $+$, we have $((\alpha \cdot x) \odot (\alpha \cdot y)^*) \oplus \alpha \cdot y = ((\alpha \cdot x) \odot (\alpha \cdot y)^*) + \alpha \cdot y$. Thus we obtain

$$((\alpha \cdot x) \odot (\alpha \cdot y)^*) + \alpha \cdot y = \alpha \cdot x \vee \alpha \cdot y \leq \alpha \cdot (x \vee y) = \alpha \cdot (x \odot y^* + y) = \alpha \cdot (x \odot y^*) + \alpha \cdot y.$$

Since $+$ is cancellative, the desired inequality is straightforward.
(e) Using (d) and Lemma 2.16 (ii), we get

$$\alpha \cdot (x \oplus y) \odot (\alpha \cdot x)^* \leq \alpha \cdot ((x \oplus y) \odot x^*) = \alpha \cdot (x^* \wedge y) \leq \alpha \cdot y.$$

It follows that

$$\alpha \cdot (x \oplus y) = \alpha \cdot (x \oplus y) \vee \alpha \cdot x = \alpha \cdot (x \oplus y) \odot (\alpha \cdot x)^* \oplus \alpha \cdot x \leq \alpha \cdot y \oplus \alpha \cdot x.$$

(f) Since $(\alpha \cdot x) \odot (\alpha \cdot y)^* \leq (\alpha \cdot y)^* \oplus (\alpha \cdot x) = ((\alpha \cdot y) \odot (\alpha \cdot x)^*)^*$, by definition partial addition $+$, $((\alpha \cdot x) \odot (\alpha \cdot y)^*) + ((\alpha \cdot y) \odot (\alpha \cdot x)^*)$ is defined and by part (d), we obtain $d(\alpha \cdot x, \alpha \cdot y) = ((\alpha \cdot x) \odot (\alpha \cdot y)^*) \oplus ((\alpha \cdot y) \odot (\alpha \cdot x)^*) = ((\alpha \cdot x) \odot (\alpha \cdot y)^*) + ((\alpha \cdot y) \odot (\alpha \cdot x)^*) \leq \alpha \cdot (x \odot y^*) + \alpha \cdot (y \odot x^*) = \alpha \cdot ((x \odot y^*) + (y \odot x^*)) = \alpha \cdot ((x \odot y^*) \oplus (y \odot x^*)) = \alpha \cdot d(x, y)$ □

For a nonempty subset N of A, the smallest $\cdot$-ideal of A which contain N, i.e., $\bigcap\{I \in Id_p(A) : N \subseteq I\}$, is said to be the $\cdot$-ideal of A generated by N and will be denoted by $(N]^{\cdot}$.

Proposition 3.3. *Let A be a PMV-algebra.*
(i) If $N \subseteq A$ is a nonempty set, then we have $(N]^{\cdot} = \{x \in A : x \leq x_1 \oplus \ldots \oplus x_n \oplus \alpha_1 \cdot y_1 \oplus \ldots \oplus \alpha_m \cdot y_m \quad \text{for some} \quad x_1, \ldots, x_n, y_1, \ldots y_m \in N, \alpha_1, \ldots \alpha_m \in A\}$, where by $(N]^{\cdot}$.
In particular, for $a \in A$,

$$(a]^{\cdot} = \{x \in A : x \leq na \oplus m(\alpha \cdot a) \quad \text{for some integers} \quad n, m \geq 0, \alpha \in A\},$$

(ii) If $I_1, I_2 \in Id_p(A)$, then
$I_1 \vee I_2 = (I_1 \cup I_2]^{\cdot} = \{a \in A : a \leq x_1 \oplus x_2 \quad \text{for some} \quad x_1 \in I_1 \quad \text{and} \quad x_2 \in I_2\}$,

(iii) If $x, y \in A$, then $(x \wedge y]^{\cdot} \subseteq (x]^{\cdot} \cap (y]^{\cdot}$.

Proof. (i) We denote $I = \{x \in A : x \leq x_1 \oplus \ldots \oplus x_n \oplus \alpha_1 \cdot y_1 \oplus \ldots \oplus \alpha_m \cdot y_m$ for some $x_1, \ldots, x_n, y_1, \ldots y_m \in N, \alpha_1, \ldots \alpha_m \in A\}$ and prove that I is the smallest $\cdot$-ideal containing N. It is clear that $N \subseteq I$, if $x \in N$, then $x \in A$, $x \leq x \oplus 0 \cdot x$ for some $x \in N, 0 \in A$, hence, $x \in I$. Let $a \leq b$ and $b \in I$. So there exist $x_1, \ldots x_n, y_1, \ldots, y_m \in N$ and $\alpha_1, \ldots, \alpha_m \in A$ such that $a \leq b \leq x_1 \oplus \ldots \oplus x_n \oplus \alpha_1 \cdot y_1 \oplus \ldots \oplus \alpha_m \cdot y_m$. It follows that $a \in I$. Now, let $a, b \in I$. Then $a \leq x_1 \oplus \ldots \oplus x_n \oplus \alpha_1 \cdot y_1 \oplus \ldots \oplus \alpha_m \cdot y_m$ for some $x_1, \ldots, x_n, y_1, \ldots y_m \in N, \alpha_1, \ldots \alpha_m \in A$, and $b \leq t_1 \oplus \ldots \oplus t_k \oplus \beta_1 \cdot z_1 \oplus \ldots \oplus \beta_s \cdot z_s$ for some $t_1, \ldots, t_k, z_1, \ldots z_s \in N$ and $\beta_1, \ldots \beta_s \in A$, by Lemma 2.2 (8), we have $a \oplus b \leq x_1 \oplus \ldots \oplus x_n \oplus t_1 \oplus \ldots \oplus t_k \oplus \alpha_1 \cdot y_1 \oplus \ldots \oplus \alpha_m \cdot y_m \oplus \beta_1 \cdot z_1 \oplus \ldots \oplus \beta_s \cdot z_s$, so $a \oplus b \in I$. Let $\alpha \in A$, $x \in I$. Then $x \leq x_1 \oplus \ldots \oplus x_n \oplus \alpha_1 \cdot y_1 \oplus \ldots \oplus \alpha_m \cdot y_m$ for some $x_1, \ldots, x_n, y_1, \ldots y_m \in N$, $\alpha_1, \ldots \alpha_m \in A$, by Lemma 2.16 (ii) and Lemma 3.2 (e), we have $\alpha \cdot x \leq \alpha \cdot x_1 \oplus \ldots \oplus \alpha \cdot x_n \oplus (\alpha \cdot \alpha_1) \cdot y_1 \oplus \ldots \oplus (\alpha \cdot \alpha_m) \cdot y_m$ for some $x_1, \ldots, x_n, y_1, \ldots y_m \in N$, $\gamma_1, \ldots \gamma_m \in A$ such that $\gamma_i = \alpha \cdot \alpha_i, i = 1, \ldots m$. Hence, $\alpha \cdot x \in I$. Thus, I is an $\cdot$-ideal containing N. Let K be another $\cdot$-ideal of A that contains N and $a \in I$. Hence, $a \leq x_1 \oplus \ldots \oplus x_n \oplus \alpha_1 \cdot y_1 \oplus \ldots \oplus \alpha_m \cdot y_m$ for some $x_1, \ldots, x_n, y_1, \ldots y_m \in N$, $\alpha_1, \ldots \alpha_m \in A$. Since K is an $\cdot$-ideal, it follows that $x_1 \oplus \ldots \oplus x_n \in K$ and $\alpha_i \cdot y_i \in K$ for $i = 1 \ldots m$. Hence, $x_1 \oplus \ldots \oplus x_n \oplus \alpha_1 \cdot y_1 \oplus \ldots \oplus \alpha_m \cdot y_m \in K$, so $a \in K$, we deduce that $I \subseteq K$. Therefore $(N]^{\cdot} = I$.
Clearly, for $a \in A$

$$(a]^{\cdot} = \{x \in A : x \leq na \oplus m(\alpha \cdot a) \quad \text{for some integers} \quad n, m \geq 0\}.$$

(ii) Follows by (i).
(iii) Suppose that $t \in (x \wedge y]$. Hence we have $t \leq n(x \wedge y) \oplus m(\alpha \cdot (x \wedge y))$, for some integers $n, m \geq 0$, $\alpha \in A$. Since $x \wedge y \leq x$, so by Lemma 2.16 (ii), we get $\alpha \cdot (x \wedge y) \leq \alpha \cdot x$. Hence $t \leq nx \oplus m(\alpha \cdot x)$. Thus $t \in (x]$. □

Following example shows equality of part (iii) of Proposition 3.3 does not hold, in general.

Example 3.4. Let $G = \oplus\{Z_i\}_{i\in\mathbb{N}}$ be the lexicographic product of denumerable infinite copies of the abelian l-group $\mathbb{Z}$ of the relative integers and $e^i \in G$ such that $e^i_k = 0$ if $k \neq i$ and $e^i_k = 1$ if $k = i$, then G with the usual product is an lu-ring. It follows that $A = \Gamma(G, u) = [0, u]$ is a PMV-algebra, where Γ is a functor from the category of abelian lu-ring to the category PMV-algebras and $u = (1, 0, 0, 0, \ldots)$ is the strong unit of A, where $\leq$ is the lexicographic order on G. If we set $P_i = ((0, e^i)]$, then $P_i \subseteq P_j$, for $i > j$ [9]. We have $((0, e^1)] \cap ((0, e^2)] = ((0, e^2)] \nsubseteq ((0, e^1) \wedge (0, e^2)] = ((0, 0, 0, \cdots 0)]$

In the above theorem, if we consider A as a unital PMV-algebra, then we have:

Corollary 3.5. *Let A be a unital PMV-algebra. If $N \subseteq A$ is a nonempty set, then we have:*
(i) $(N]^{\cdot} = \{x \in A : x \leq x_1 \oplus \ldots \oplus x_n \quad \textit{for some} \quad x_1, \ldots, x_n \in N\} = (N]$, *In particular, for $a \in A$,*

$$(a]^{\cdot} = (a] = \{x \in A : x \leq na \quad \textit{for some integer} \quad n \geq 0\},$$

(ii) If $I_1, I_2 \in Id_p(A)$, then $I_1 \vee I_2 = (I_1 \cup I_2]^{\cdot} = (I_1 \cup I_2] = \{a \in A : a \leq x_1 \oplus x_2 \quad \textit{for some} \quad x_1 \in I_1, x_2 \in I_2\}$,
(iii) If $x, y \in A$, then $(x \wedge y] = (x] \cap (y]$.

For $I \in Id_p(A)$ and $a \in A \setminus I$, we denote by $I(a) = (a]^{\cdot} \vee I = (I \cup \{a\}]^{\cdot}$.

Remark 3.6. Let A be a PMV-algebra. Then
$I(a) = \{x \in A : x \leq y \oplus ma \oplus n(\alpha \cdot a), \quad \text{for some} \quad y \in I, \quad \text{integers} \quad n, m \geq 0, \alpha \in A\}$.

Proof. Let $T = \{x \in A : x \leq y \oplus ma \oplus n(\alpha \cdot a), \text{for some } y \in I, \text{ integers } n, m \geq 0, \alpha \in A\}$. We suppose that, $x \in I(a) = (a]^{\cdot} \vee I = \{x \in A : x \leq x_1 \oplus y \quad \text{for some} \quad x_1 \in (a]^{\cdot} \quad \textit{and} \quad y \in I\}$. Since $x_1 \in (a]^{\cdot}$, then $x_1 \leq ma \oplus n(\alpha \cdot a)$, for some integers $m, n \geq 0$ and $\alpha \in A$, we have $x \leq x_1 \oplus y \leq ma \oplus n(\alpha \cdot a) \oplus y$, it follows that $x \in T$.
Conversely, if $x \in T$, then we get that $x \leq y \oplus ma \oplus n(\alpha \cdot a)$, for some $y \in I$ and integer $n \geq 0, x_1 = ma \oplus n(\alpha \cdot a) \in (a]^{\cdot}$, so $x \leq y \oplus x_1$ such that $x_1 \in (a]^{\cdot}$ and $y \in I$. It follows that $x \in (a]^{\cdot} \vee I = I(a)$.

□

Remark 3.7. Let A be a unital PMV-algebra. We have: $I(a) = \{x \in A : x \leq y \oplus na, \quad \text{for some} \quad y \in I \text{ and an integer } n \geq 0\}$.

Corollary 3.8. *Let $I \in Id_p(A)$ and $a, b \in A \setminus I$. Then $I(a \wedge b) \subseteq I(a) \cap I(b)$.*

Proof. We have $a \wedge b \leq y \oplus m(a \wedge b) \oplus n(\alpha \cdot (a \wedge b))$ for some $y \in I$ and integers $m, n \geq 0$. Then $a \wedge b \in I(a \wedge b)$. Since $a \wedge b \leq a, b$ and $a \in (a]^{\cdot}, \quad b \in (b]^{\cdot}$, so $a \wedge b \in (a]^{\cdot} \subseteq I(a)$ and $a \wedge b \in (b]^{\cdot} \subseteq I(b)$, also $I \subseteq I(a)$, $I \subseteq I(b)$, we deduce that $a \wedge b \in I(a) \cap I(b)$, if $x \in I(a \wedge b)$, then $x \leq y \oplus m(a \wedge b) \oplus n(\alpha \cdot (a \wedge b))$. It follows that $x \in I(a) \cap I(b)$, Thus $I(a \wedge b) \subseteq I(a) \cap I(b)$. □

We note that if A is a unital PMV-algebra, then any ideal of A is an $\cdot$-ideal. Thus the ideals and the $\cdot$-ideals of A coincide.

Corollary 3.9. *Let A be a unital PMV-algebra, $I \in Id_p(A)$ and $a, b \in A \backslash I$. Then $I(a \wedge b) = I(a) \cap I(b)$.*

Proof. Since A is a unital PMV-algebra, by the above notation, it is clear that I is an ideal of an MV-algebra A, so $I(a \wedge b) = I(a) \cap I(b)$, [11]. □

We recall that if M and N are two PMV-algebras, then a PMV-algebra homomorphism is an MV-algebra homomorphism $h : M \to N$ such that $h(x \cdot y) = h(x) \cdot h(y)$, for any $x, y \in M$.

Also $ker(h) = \{x \in M : h(x) = 0\}$ is an $\cdot$-ideal of M.

Lemma 3.10. *Let M, N be PMV-algebras and $f : M \longrightarrow N$ be a PMV-algebra homomorphism. Then for each $\cdot$-ideal $J \in Id_p(N)$, the set $f^{-1}(J) = \{x \in M : f(x) \in J\}$ is an $\cdot$-ideal of M. Thus, in particular, $ker(f) \in Id_p(M)$.*

The following remarks are similar to Theorems 2.24 and 2.25 and Propositions 2.21 and 2.22 of [13].

Remark 3.11. (The first isomorphism theorem) If M and N are two PMV-algebras and $f : M \longrightarrow N$ is a PMV-algebra homomorphism, then $M/ker(f)$ and $Im(f)$ are isomorphic PMV-algebras.

Remark 3.12. (The second isomorphism theorem) If M is a PMV-algebra and I, J are two $\cdot$-ideals such that $I \subseteq J$, then $(M/I)/P_I(J)$ and M/J are isomorphic PMV-algebra, such that $P_I : M \to M/I$ is the quotient of M.

We recall that we have distance function $d : A \times A \to A$ is defined by $d(x, y) = (x \odot y^*) \oplus (y \odot x^*)$ [13].
We note that if A be a PMV-algebra and $I \subseteq A$ an $\cdot$-ideal of A, then the relation $\sim_I$ defined by $x \sim_I y$ if and only if $d(x, y) \in I$.

For any $x, y \in A$, $\sim$ is a congruence with respect to the PMV-algebra operations. Using Lemma 3.2 (f), we infer that

$$x \sim_I y \text{ implies } \alpha \cdot x \sim_I \alpha \cdot y,$$

for any $\alpha \in A$. Thus, the quotient PMV-algebra A/I has a canonical structure of PMV-algebra

$$\alpha \cdot [x]_I := [\alpha \cdot x]_I,$$

where $[x]_I$ is the congruence class of x.

Remark 3.13. If $\sim$ is a congruence relation on A, then $I_\sim = \{x \in A : \quad x \sim 0\} \in Id_p(A)$ and $x \sim y$ if and only if $d(x, y) \sim 0$ (Since $d(x, 0) = (x \odot 0^*) \oplus (0 \odot x^*) = x$ Thus $d(x, 0) = x$ (1). Also, we have $x \sim y$ iff $d(x, y) \in I$, for $\cdot$-ideal I of A. Now, if $x \sim y$, then by (1) we obtain $d(d(x, y), 0) = d(x, y) \in I$, so we get $d(x, y) \sim 0$ and converse is hold.)

Remark 3.14. Let I be an $\cdot$-ideal of A and $\sim$ be a congruence relation on A. The assignment $I \rightsquigarrow \sim_I$ is a bijection from the set $Id_p(A)$ of $\cdot$-ideals of A onto the set of congruences on A; more precisely, the function $\alpha : Id_p(A) \to Con(A)$ defined by $\alpha(I) = \sim_I$ is an isomorphism of partially ordered sets.

4. Product MV-algebra of fractions relative to a $\cdot$-closed system

Definition 4.1. A nonempty subset of a PMV-algebra S of A is called $\cdot$-closed system in A if $1 \in S$, and if $x, y \in S$, then $x \cdot y \in S$.

We denote by $S(A)$ the set of all $\cdot$-closed systems of A (clearly $\{1\}$, $A \in S(A)$).

For $S \in S(A)$, on the PMV-algebra A we consider the relation θ_S defined by

$$(x, y) \in \theta_S \text{ iff there exists } e \in S \cap B(A) \text{ such that } x \cdot e = y \cdot e.$$

Example 4.2. Let $\Omega = \{1, 2\}$ and $\mathcal{A} = \mathcal{P}(\Omega) = \{\{1\}, \{2\}, \{1, 2\}, \emptyset\}$. $\mathcal{A}$ is a PMV-algebra with $\oplus = \cup$ and $\odot = \cdot = \cap$. The $\cdot$-closed systems of A which contain $\emptyset$ are:

(i) $S = \{A, B(A) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}, \{\emptyset, \{1\}, \{1, 2\}\}$,
$\{\emptyset, \{2\}, \{1, 2\}\}, \{\emptyset, \{1, 2\}\}\}$.

(ii) The $\cdot$-closed systems of A which do not contain $\emptyset$ are:
$S = \{\{1, 2\}, \{\{1\}, \{1, 2\}\}, \{\{2\}, \{1, 2\}\}\}$.

Example 4.3. Consider MV-algebra $A = \{0, 1/2, 1\}$ with operations $x \oplus y = 1 \wedge (x + y)$ and $x^* = 1 - x$. Then A is a PMV-algebra with the usual product of elements of A.

Case 1. $0 + 1$ is a defined, since $0 \leq 1^* = 0$. Then $0 \cdot 1 + 1 \cdot 1 = 0 + 1$ is a defined and we have $(0 + 1) \cdot 1 = 1 = 0 \cdot 1 + 1 \cdot 1$ and $0 \cdot 1/2 + 1 \cdot 1/2 = 0 + 1/2$ is defined and $(0 + 1) \cdot 1/2 = 1/2 = 0 \cdot 1/2 + 1 \cdot 1/2$.

Case 2. $1/2 + 1/2$ is a defined, since $1/2 \leq (1/2)^* = 1/2$. Then $1 \cdot 1/2 + 1 \cdot 1/2 = 1/2 + 1/2$ is a defined and we have $(1/2 + 1/2) \cdot 1 = 1 = 1/2 \cdot 1 + 1/2 \cdot 1$. Also $1/2 \cdot 1/2 + 1/2 \cdot 1/2 = 1/4 + 1/4$ is a defined, since $1/4 \leq (1/4)^* = 3/4$ and $(1/2 + 1/2) \cdot 1/2 = 1/2 = 1/2 \cdot 1/2 + 1/2 \cdot 1/2$. The other cases are clear. obviously, $S = \{1/2, 1\}$ is $\cdot$-closed of A.

Lemma 4.4. *Let A be finite PMV-algebra. Then θ_S is a congruence on A.*

Proof. The reflexivity (since $1 \in S \cap B(A)$) and the symmetry of θ_S are immediate. To prove the transitivity of θ_S, let $(x, y), (y, z) \in \theta_S$.

Thus there exists $e, f \in S \cap B(A)$ such that $x \cdot e = y \cdot e$ and $y \cdot f = z \cdot f$. We denote $g = e \cdot f$, by Theorem 2.15 and Lemma 2.2 (9), we have $e \cdot f = e \wedge f = e \odot f$, hence $g = e \cdot f \in S \cap B(A)$, also we have $g \cdot x = (e \cdot f) \cdot x = (e \cdot x) \cdot f = (y \cdot e) \cdot f = (y \cdot f) \cdot e = (z \cdot f) \cdot e = z \cdot g$, hence $(x, z) \in \theta_S$. To prove the compatibility of θ_S with the operation $+$, $*$ and $\cdot$. Let $x, y, z, t \in A$ such that $(x, y) \in \theta_S$ and $(z, t) \in \theta_S$.

Thus there exists $e, f \in S \cap B(A)$ such that $x \cdot e = y \cdot e$ and $z \cdot f = t \cdot f$, we denote $g = e \cdot f \in S \cap B(A)$.

If $x + z$ is defined in A, then $x \leq z^*$, hence by Lemma 2.16 (ii) and Lemma 3.2 (b) we have, $x \cdot g \leq z^* \cdot g \leq (z \cdot g)^*$. Which imply that $x \cdot g + z \cdot g$ is defined. It follows that $(x + z) \cdot g = (x \cdot g) + (z \cdot g) = (x \cdot e \cdot f) + (z \cdot f \cdot e) = (y \cdot e \cdot f) + (t \cdot f \cdot e) = (y \cdot g) + (t \cdot g) = (y + t) \cdot g$, hence $(x + z, y + t) \in \theta_S$.

From $x \cdot e = y \cdot e$, we deduce $(x \cdot e)^* = (y \cdot e)^*$, it follows from Lemma 3.2 (c) that $x^* \cdot e + (1 \cdot e)^* = y^* \cdot e + (1 \cdot e)^*$. Since $+$ is a canncellative operation, $x^* \cdot e = y^* \cdot e$. Hence $(x^*, y^*) \in \theta_S$.

Also if denote $g = e \cdot f \in S \cap B(A)$, then $(x \cdot z) \cdot g = (x \cdot e) \cdot (z \cdot f) = (y \cdot e) \cdot (t \cdot f) = (y \cdot t) \cdot g$. Hence $(x \cdot z, y \cdot t) \in \theta_S$. $\square$

For x we denote by x/S the equivalence class of x relative to θ_S and by

$$A[S] = A/\theta_S.$$

By $p_S : A \to A[S]$ we denote the canonical map defined by $p_S(x) = x/S$, for every $x \in A$. Clearly, in $A[S]$, $\mathbf{0} = 0/S$, $\mathbf{1} = 1/S$ and for every $x, y \in A$,

$$x/S + y/S = (x+y)/S, \quad x/S \cdot y/S = (x \cdot y)/S$$

$$\text{and } (x/S)^* = x^*/S, x/S \oplus y/S = (x \oplus y)/S.$$

So, p_S is an epimorphism of PMV-algebras.

Remark 4.5. Let A be a PMV-algebra.
Since for every $s \in S \cap B(A)$, it follows from Lemma 2.15 and Lemma 2.2 (9) that $s \wedge s = s \cdot s = s \cdot 1 = s \wedge 1$, we conclude that $s/S = 1/S = \mathbf{1}$, hence $p_S(S \cap B(A)) = \{\mathbf{1}\}$.

Proposition 4.6. *If $a \in A$ and A is a Boolean algebra, then $a/S \in B(A[S])$ if and only if there exists $e \in S \cap B(A)$ such that $e \cdot a \in B(A)$.*

Proof. For $a \in A$, we have $a/S \in B(A[S])$ if and only if $a/S \oplus a/S = a/S$ if and only if there exists $e \in S \cap B(A)$ such that $(a + a) \cdot e = a \cdot e + a \cdot e = a \cdot (e + e) = a \cdot e \in B(A)$. □

Remark 4.7. If $f : A \to B$ is a PMV-homomorphism and S is a $\cdot$-closed system of B, then $f^{-1}(S)$ is a $\cdot$-closed system.

Proof. Let $x, y \in f^{-1}(S)$. Then $f(x) \in S$ and $f(y) \in S$. Since S is a $\cdot$-closed system of B, so $f(x) \cdot f(y) \in S$. Hence $f(x \cdot y) \in S$. So $x \cdot y \in f^{-1}(S)$. Thus $f^{-1}(S)$ is a $\cdot$-closed system of A. □

Theorem 4.8. *If $A^{'}$ is a unital PMV-algebra and $f : A \to A'$ is a morphism of PMV-algebras such that $f(S \cap B(A)) = \{1\}$, then there exists a unique morphism of PMV-algebras $f' : A[S] \to A'$ such that the diagram*

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[d, "p_S"'] & & A' \\ A[S] \arrow[urr, "f'"'] & & \end{tikzcd}$$

is commutative (i.e., $f' \circ p_S = f$).

Proof. If $x, y \in A$ and $p_S(x) = p_S(y)$, then $(x, y) \in \theta_S$, hence there exists $e \in S \cap B(A)$ such that $x \cdot e = y \cdot e$. Since f is morphism of PMV-algebras, we obtain that $f(x \cdot e) = f(y \cdot e) \Leftrightarrow f(x) \cdot f(e) = f(y) \cdot f(e) \Leftrightarrow f(x) \cdot 1 = f(y) \cdot 1 \Leftrightarrow f(x) = f(y)$.

From this observation we conclude that the map $f' : A[S] \to A'$ defined for $x \in A$ by $f'(x/S) = f(x)$ is correctly defined. Clearly, f' is a morphism of PMV-algebras.

The uniqueness of f' follows from the fact that p_S is an onto map. □

Remark 4.9. The previous theorem allows us to call $A[S]$ the PMV-algebra of fractions relative to the $\cdot$-closed system S.

Theorem 4.10. *If I is an $\cdot$-ideal of PMV-algebra A, $\{0\}$ is an $\cdot$-prime ideal of A and S is a $\cdot$-closed system such that $0 \notin S$, then $I(S) = S^{-1}(I) = \{x/S : x \in I\}$ is an $\cdot$-ideal of $A[S]$.*

Proof. (1) Let $a/S, b/S \in S^{-1}(I)$. Then $a, b \in I$. Since I is an $\cdot$-ideal of A, $a \oplus b \in I$. Hence

$$a/S \oplus b/S = (a \oplus b)/S \in S^{-1}(I).$$

(2) If $a/S \leq x/S$ and $x/S \in S^{-1}(I)$, hence $a/S \odot (x/S)^* = 0/S$. We get that $(a \odot x^*)/S = 0/S$, then $e \cdot (a \odot x^*) = 0 \cdot e = 0 \in \{0\}$. Since $e \in S \cap B(A)$, and $e \neq 0$ and $\{0\}$ is $\cdot$-prime ideal, hence $a \odot x^* = 0$, so $a \leq x \in I$, thus $a \in I$. It follows that $a/S \in S^{-1}(I)$.
(3) If $x/S \in A[S]$ and $a/S \in S^{-1}(I)$, we show that $x/S \cdot a/S \in S^{-1}(I)$.
Hence by $\cdot$-ideal property, we have $x/S \cdot a/S = (x \cdot a)/S \in S^{-1}(I)$. Thus $S^{-1}(I)$ is an $\cdot$-ideal of $A[S]$. □

Remark 4.11. If P is an obstinate ideal of A, $\{0\}$ is an $\cdot$-prime ideal of A and S is a $\cdot$-closed system such that $0 \notin S$, then $S^{-1}(P)$ is an obstinate ideal of $A[P]$ (Since P is an obstinate ideal of A, by Lemma 2.7, it follows that $x \in P$ or $x^* \in P$ if and only if $x/S \in S^{-1}(P)$ or $x^*/S = (x/S)^* \in S^{-1}(P)$. So $S^{-1}(P)$ is an obstinate ideal of $A[S]$.)

Theorem 4.12. *Let A be a unital PMV-algebra, S be a $\cdot$-closed system, P and $\{0\}$ be $\cdot$-prime ideals of A such that $P \cap S = \emptyset$. Then $S^{-1}(P)$ is an $\cdot$-prime ideal of $A[S]$.*

Proof. Since P is an $\cdot$-prime ideal of A, $1 \notin P$, so $1/S \notin S^{-1}(P)$. For any $x/S, y/S \in S^{-1}(P)$, $x/S \cdot y/S \in S^{-1}(P)$. This results $(x \cdot y)/S \in S^{-1}(P)$. Thus $(x \cdot y)/S = t/S$ such that $t \in P$. So there exists $e \in B(A) \cap S$ such that $e \cdot (x \cdot y) = e \cdot t \leq 1 \cdot t = t \in P$, (since $e \notin P$, $x \cdot y \in P$. Also by hypothesis, we conclude that $x \in P$ or $y \in P$. Thus $x/S \in S^{-1}(P)$ or $y/S \in S^{-1}(P)$. □

Remark 4.13. Consider A is a unital PMV-algebra.

1. If $S = \{1\}$ or S is such that $1 \in S$ and $S \cap (B(A) \setminus \{1\}) = \emptyset$, then for $x, y \in A$, $(x, y) \in \theta_S \Longleftrightarrow x \cdot 1 = y \cdot 1 \Longleftrightarrow x = y$, hence in this case $A[S] = A$.
2. If S is a $\cdot$-closed system such that $0 \in S$ (for example $S = A$ or $S = B(A)$), then for every $x, y \in A$, $(x, y) \in \theta_S$ (since $x \cdot 0 = y \cdot 0$ and $0 \in S \cap B(A)$), hence in this case $A[S] = \mathbf{0}$.
3. If P is an $\cdot$-prime ideal of A, then $S = A \setminus P$ is a $\cdot$-closed system. We denote $A[S]$ by A_P. The set $M = S^{-1}(P) = \{x/S : x \in P\}$ is a maximal $\cdot$-ideal of A_P. Indeed, if $x, y \in P$, then $x/S \oplus y/S = (x \oplus y)/S \in M$ (since $x \oplus y \in P$). If $x, y \in A$ such that $x \in P$ and $y/S \leq x/S$, then $(y^* \oplus x)/S = 1/S$, hence there exists $e \in S \cap B(A)$ such that $e = e \cdot 1 = e \cdot (y^* \oplus x) \leq (y^* \oplus x) \cdot 1 = y^* \oplus x$, so $e \leq y^* \oplus x$, It follows from Lemma 2.2 (10) that $e \odot y \leq x$. Also by Lemma 2.14 and Lemma 2.2 (9), we conclude that $e \cdot y \leq e \wedge y = e \odot y \leq x$. Since $x \in P$, $y \odot e \in P$, so $y \cdot e \in P$. Hence $y \in P$ (since $e \notin P$), so $y/S \in M$. Also, if $a/S \in A[S]$ and $x/S \in M$ such that $x \in P$, then $a/S \cdot x/S = (a \cdot x)/S \in M$ (Since $a \cdot x \in P$). To prove the maximality of M, let I be an $\cdot$-ideal of A_P such that $M \subseteq I$ and $M \neq I$. Then there exists $x/S \in I$ such that $x/S \notin M$, (that is $x \notin P \Longleftrightarrow x \in S$), hence $x/S = \mathbf{1}$, so $I = A_P$. Moreover, M is the only maximal $\cdot$-ideal of A_P (since if we have another maximal $\cdot$-ideal M' of A_P, then $M' \nsubseteq M$, hence there exists $x/S \in M'$ such that $x/S \notin M$, so $x/S = \mathbf{1}$ and $M' = A_P$, which is a contradiction.)

In other words A_P is a local PMV-algebra. The process of passing from A to A_P is called a localization at P.

Example 4.14. Consider Example 4.2.
(i) If $S = \{A, B(A) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}, \{\emptyset, \{1\}, \{1, 2\}\}$,

$\{\emptyset, \{2\}, \{1,2\}\}, \{\emptyset, \{1,2\}\}\}$.

In all these cases $A[S] = \mathbf{0}$.

(ii) If $S = \{\{1,2\}, \{\{1\}, \{1,2\}\}, \{\{2\}, \{1,2\}\}\}$. In all these cases $A[S] = A$ because $S \cap B(A) = \{1,2\}$.

5. Conclusions

The notions of MV-algebras and ideals of MV-algebras are defined by C. C. Chang [2]. Then A. Dvurečenskij and A. Di Nola in [4] introduced the notion of product MV-algebras, that is MV-algebras whose product operation $(\cdot)$ is defined on the whole MV-algebra. We investigated $\cdot$-ideals in product MV-algebras, we studied the $\cdot$-ideals of a product MV-algebra A generated by a subset of A and we introduced the notion of $\cdot$-closed system of a product MV-algebra A. Then we introduced the notion of product MV-algebra of fractions relative to a $\cdot$-closed system.

Acknowledgments The authors thanks the referees for their valuable comments and suggestions.

References

[1] H. banivaheb and A. borumand Saeid, *Localization of PMV-algebras*, Soft Computing, **22** (2018), 31–40.

[2] C. C. Chang, *Algebraic analysis of many valued logic*, Trans. Amer. Math. Soc., **88** (1958), 467–490.

[3] R. Cignoli, I. M. L. D'Ottaviano, D. Mundici, *Algebraic Foundations of Many-Valued Reasoning*, Kluwer Academic, Dordrecht, (2000).

[4] A. Di Nola, A. Dvurečenskij, *Product MV-algebras*, Multiple-Valued Logics **6** (2001), 193–215.

[5] A. Di Nola, P. Flondor, and I. Leuştean, *MV-modules*, J. Algebra, **267** (2003), 21–40.

[6] A. Dvurečenskij, *On partial addition in pseudo MV-algebras*, in: Proceedings of the Fourth International Symposium on Economic Informatics, 1999, 952–960.

[7] A. Filipoiu, G. Georgescu, A. Lettieri, *Maximal MV-algebras*, Mathware, Soft Computing, **4** (1997), 53–62.

[8] J. M. Font, A. J. Rodriguez, A. Torrens, *Wajsberg algebras*, Stochastica **8** (1984), 5–31.

[9] F. Forouzesh, E. Eslami, A. Borumand Saeid, *On prime A-ideals in MV-modules*, University Politehnica of Bucharest Scientific Bulletin **87** (2014), 181–198.

[10] F. Forouzesh, E. Eslami, A. Borumand Saeid, *Radical of A-ideals in MV -modules*, Analeles, Tiint, Ifice Ale Universitat, II "AL.I. Cuza" Dinias, I (S. N.) Matematica, Tomul Lexii, 2016, f.1, 33–56.

[11] F. Forouzesh, E. Eslami, A. Borumand Saeid, *On obstinate ideals in MV-algebras*, Politehn. Univ. Bucharest. Sci Bull Ser. Appli Math. Phys. **76** (2014), 53–62.

[12] A. Iorgulescu, *Algebras of logic as BCK algebras*, Academy of Economic Studies Bucharest, Romania, (2008).

[13] D. Piciu, *Algebras of fuzzy logic*, Ed. Universitaria Craiova (2007).

[14] S. Saidi Goraghani and R. A. Borzooei, *On injective MV-modules*, Bulletin of the Section of Logic, **4** (2018), 283–298.