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GRADED S-CO-r-SUBMODULES AND GRADED S-CO-n-SUBMODULES

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ABSTRACT

Let G be a group with identity element e, and let $R = \bigoplus_{g\in G} R_g$ be a commutative graded ring. Let M be a graded R-module, and let $S \subseteq h(R)$ be a multiplicatively closed subset of R. The aim of this paper is to introduce and study the concepts of graded S-co-r-submodules and graded S-co-n-submodules. We investigate some of their properties and establish the relationship between graded S-co-r-submodules and graded S-co-n-submodules.

1. Introduction

The study of graded rings arises naturally from the theory of affine schemes and provides a powerful framework for formalizing and unifying inductive arguments. However, this is not merely an algebraic technique. The concept of grading in algebra, particularly that of graded modules, is essential in the study of the homological aspects of modules. Much of the modern development of commutative algebra places significant emphasis on graded rings, which play a central role in both commutative algebra and algebraic geometry. Gradings occur in a wide variety of contexts, at both elementary and advanced levels. In recent years, rings endowed with a group-graded structure have become increasingly important; consequently, graded

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analogues of many classical concepts have been extensively investigated (see [1]-[7] and [10]). In this paper, we introduce and study the notions of graded S-co-r-submodules and graded S-co-n-submodules of graded rings, and we investigate several fundamental properties of these graded submodules.

Let G be a group with identity element e, and let R be a ring. The ring R is said to be a G-graded if $R = \bigoplus_{g\in G} R_g$ such that $R_gR_h \subseteq R_{gh}$ for all $g, h \in G$, where R_g is an additive subgroup of R for all $g \in G$. The elements of R_g are homogeneous of degree g. An element r of R has a unique decomposition as $r = \sum_{g\in G} r_g$ with $r_g \in R_g$ for all $g \in G$, but the sum being a finite sum i.e. almost all r_g zero. Let $R = \bigoplus_{g\in G} R_g$ be a graded ring. A subring S of R is called a graded subring of R if $S = \sum_{g\in G}(R_g \cap S)$. Equivalently, S is graded if for every element $x \in S$ all the homogeneous components of x (as an element of R) are in S. Let $R = \bigoplus_{g\in G} R_g$ be a graded ring. Then $1 \in R_e$ and R_e is a subring of R. Let I be an ideal of a graded ring R. Then I is said to be a graded ideal of R, if $I = \bigoplus_{g\in G}(I \cap R_g)$, i.e., for $x \in I$, $x = \sum_{g\in G} x_g$, where $x_g \in I$ for all $g \in G$. Moreover, R/I becomes a G-graded ring with g-component $(R/I)_g = (R_g + I)/I$ for $g \in G$. Let I be a graded ideal of R. Then the graded radical of I is denoted by $Grad(I)$ and defined by $Grad(I) = \{r \in R :$ for any $g \in G$, $r_g^{n_g} \subseteq I$ for some $n_g \in \mathbb{N}\}$. A proper graded ideal P of R is called graded prime, if $a_gb_h \in P$ for some $a_g, b_h \in h(R)$, then $a_g \in P$ or $b_h \in P$ [11]. We call $S \subseteq h(R)$ is a multiplicatively closed subset of R if (i) $0 \notin S$, (ii) $1 \in S$, and (iii) $s_gs'_{g'} \in S$ for all $s_g, s'_{g'} \in S$. Let R be a graded ring and M an R-module. We say that M is a graded R-module, if there exists a family of subgroups $\{M_g\}_{g\in G}$ of M such that
(1) $M = \bigoplus_{g\in G} M_g$,
(2) $R_gM_h \subseteq M_{gh}$ for all $g, h \in G$.

The elements of M_g are called homogeneous of degree g. It is clear that M_g is an R_e-submodule of M for all $g \in G$. Moreover $h(M) = \bigcup_{g\in G} M_g$. Let $M = \bigoplus_{g\in G} M_g$ and $M' = \bigoplus_{g\in G} M'_g$ be two graded R-modules. A mapping g from M into M' is said to be a homomorphism, if for all $m, n \in M$;

(1) $g(m + n) = g(m) + g(n)$,
(2) $g(rm) = rg(m)$, for any $r \in R$ and $m \in M$,
(3) For any $g \in G$; $g(M_g) \subseteq M'_g$.

Let N be a graded submodule of a graded R-module M, the graded ideal $\{r \in R \mid rM \subseteq N\}$ will be denoted by $(N : M)$. Then $(0 : M) = Ann(M)$ is the annihilator of M. Let $S \subseteq h(R)$ be a multiplicatively closed subset of R and N a graded submodule of M with $(P :_R M) \cap S = \emptyset$. Then the graded submodule P is called a graded S-prime (resp. graded S-primary) submodule of M, if there exists $s \in S$ such that whenever $a_gm_h \in P$, then $sa_g \in (P :_R M)$ or $sm_h \in P$ ($sa_g \in Rad(P :_R M)$ or $sm_h \in P$) for each $a_g \in h(R)$ and $m_h \in h(M)$. Particularly, a graded ideal I of R is called a graded S-prime (resp. graded S-primary) ideal of R if I is a graded S-prime (resp. graded S-primary) submodule of graded R-module R see, [12]. Let M be a graded R-module. The subset $W^{gr}_R(M)$ of R is defined by $\{r \in h(R) \mid rM \neq M\}$ and $U^{gr}(R)$ is the set of the unit homogeneous elements of R. Generalizations of graded prime submodules play a fundamental role in modern module theory and its related areas. Their significance lies in providing greater flexibility for structural analysis, facilitating decomposition theory, establishing connections with algebraic geometry and spectral theory, and enabling applications to algebraic graphs and combinatorial structures. In this paper, we introduce and study the concepts of graded S-co-r-submodules and graded S-co-n-submodules as generalizations of graded prime submodule. We get some properties of them, for example, we show that every graded S-copure submodule is a graded S-co-r-submodule. Also, we show that every graded S-co-n-submodule is a graded S-co-r-submodule.

2. Graded S-co-r-submodules

Definition 2.1. (a) A non-zero graded submodule N of a graded R-module M is called a graded co-r-submodule, if $a_gN \subseteq K$ and $a_g \notin W_R^{gr}(M)$ for some $a_g \in h(R)$ and a graded submodule K of M, then $N \subseteq K$.

(b) Let $S \subseteq h(R)$ be a multiplicatively closed subset of R, and let M be a graded R-module such that $W_R^{gr}(M) \cap S = \emptyset$. A non-zero graded submodule N of M is called a graded S-co-r-submodule, if there exists an element $s \in S$ such that, whenever $a_gN \subseteq K$ and $sa_g \notin W_R^{gr}(M)$ for some $a_g \in h(R)$ and a graded submodule K of M, then $sN \subseteq K$. This fixed element $s \in S$ is called an S-element of N.

Note that if $S \cap W_R^{gr}(M) \neq \emptyset$, then there exists $s \in S$ such that $sM \neq M$. Consequently, for any $a_g \in h(R)$, we have $sa_gM \neq M$ i.e., $sa_g \in W_R^{gr}(M)$. Indeed, if $sa_{gr}M = M$, then $M = sa_{gr}M \subseteq sM$, which implies $sM = M$, a contradiction.

Remark 2.2. Let N be a non-zero graded submodule of a graded R-module M. If N is a graded co-r-submodule of M, then clearly, N is a graded S-co-r-submodule for any multiplicatively closed subset S of R. However, the classes of graded co-r-submodules and graded S-co-r-submodules coincide if $S \subseteq U^{gr}(R)$.

A graded R-module M is said to be a graded S-multiplication module, if for every graded submodule N of M, there exists $s \in S$ such that $sN \subseteq IM \subseteq N$ for some graded ideal I of R.

Theorem 2.3. *Let $S \subseteq h(R)$ be a multiplicatively closed subset of R and M be a graded S-multiplication module with $W_R^{gr}(M) \cap S = \emptyset$. Then every non-zero graded submodule N of M is a graded S-co-r-submodule.*

Proof. Let N be a non-zero graded submodule of M. Then there exists $s \in S$ such that $sN \subseteq IM \subseteq N$ for some graded ideal I of R since M is a graded S-multiplication module. Assume that $a_gN \subseteq K$ with $sa_gM = M$ for $a_g \in h(R)$ and a graded submodule K of M. Thus we have $sN \subseteq IM = I(sa_gM) = sa_g(IM) \subseteq sa_gN \subseteq sK \subseteq K$. Therefore, N is a graded S-co-r-submodule. □

Example 2.4. Consider the $\mathbb{Z}_2$ graded ring $\mathbb{Z}$ and the graded $\mathbb{Z}$-module $\mathbb{Z}$ and the multiplicatively closed subset $S = \mathbb{Z} - 2\mathbb{Z}$. Then every non-zero graded submodule of $\mathbb{Z}$ is a graded S-co-r-submodule.

Let $S \subseteq h(R)$ be a multiplicatively closed subset of R. The saturation S^* of S is defined as $S^* = \{x \in h(R) \mid \frac{x}{1} \text{ is a unit of } S^{-1}R\}$. Note that $S^* \subseteq h(R)$ is a multiplicatively closed subset of R containing S.

Proposition 2.5. *Let M be a graded R-module and $S \subseteq h(R)$ be a multiplicatively closed subset of R with $W_R^{gr}(M) \cap S = \emptyset$. Then the following statements hold:*

(i) *Let $S_1 \subseteq S_2 \subseteq h(R)$ be multiplicatively closed subsets of R. If N is a graded S_1-co-r-submodule and $W_R^{gr}(M) \cap S_2 = \emptyset$, then N is a graded S_2-co-r-submodule.*

(ii) *A graded submodule N of M is a graded S-co-r-submodule if and only if it is a graded S^*-co-r-submodule.*

Proof. The proofs are completely straightforward. □

Proposition 2.6. *Let M be a graded R-module, and let $S \subseteq h(R)$ be a multiplicatively closed subset of R such that $W_R^{gr}(M) \cap S = \emptyset$. Then the following statements hold.*

(i) *M is a graded S-co-r-submodule of M.*

(ii) *If $\{N_i\}_{i=1}^n$ is a family of graded S-co-r-submodules of M, then $\sum_{i=1}^n N_i$ is a graded S-co-r-submodule of M.*

Proof. (i) It is clear.

(ii) Let N_i be a graded S-co-r-submodule of M with S-element s_i for each $i = 1, 2, \ldots, n$. We take $s = s_1 \cdots s_n$. Assume that $a_g \sum_{i=1}^n N_i \subseteq K$ and $a_g s \notin W_R^{gr}(M)$ for some $a_g \in h(R)$ and a graded submodule K of M. Thus for each i, $a_g N_i \subseteq K$ and $s_i a_g \notin W_R^{gr}(M)$. Since N_i is a graded S-co-r-submodule, $s_i N_i \subseteq K$ and so $sN_i \subseteq K$ for each $i = 1, \ldots, n$. Therefore, $s \sum_{i=1}^n N_i \subseteq K$, as required. □

Note that the intersection of two graded S-co-r-submodules does not necessarily form a graded S-co-r-submodule.

Example 2.7. Let $G = \mathbb{Z}_2$. Then consider the G-graded ring $\mathbb{Z}$, G-graded module $M = \mathbb{Z}_6 \times \mathbb{Z}_2$ with components $M_0 = \mathbb{Z}_6 \times \{0\}$ and $M_1 = \{0\} \times \mathbb{Z}_2$ and the multiplicatively closed subset $S = \{5^n | n \in \mathbb{N} \cup \{0\}\}$ of $\mathbb{Z}$. We have $\langle(\bar{2}, \bar{0})\rangle$ and $\langle(\bar{3}, \bar{0})\rangle$ are graded S-co-r-submodules of $M = \mathbb{Z}_6 \times \mathbb{Z}_2$. But $\langle(\bar{2}, \bar{0})\rangle \cap \langle(\bar{3}, \bar{0})\rangle = 0$.

Proposition 2.8. *Let M be a graded R-module and $S \subseteq h(R)$ be a multiplicatively closed subset of R with $W_R^{gr}(M) \cap S = \emptyset$. Then if N is a graded S-co-r-submodule of M and A be a graded ideal of R with $A \nsubseteq Ann_R(N)$, then AN is a graded S-co-r-submodule of M. In particular, AM is a graded S-co-r-submodule if $A \nsubseteq AM$.*

Proof. Suppose that N is a graded S-co-r-submodule of M with S-element $s \in S$. Let $r_g AN \subseteq K$ and $sr_g M = M$ for $r_g \in h(R)$ and a graded submodule K of M. Then $r_g a_h N \subseteq K$ for every $a_h \in A \cap h(R)$, so $r_g N \subseteq (K :_M a)$. Since N is a graded S-co-r-submodule, we conclude $sN \subseteq (K :_M a_h)$ and so $sa_h N \subseteq K$ for every $a_h \in A \cap h(R)$. Thus $sAN \subseteq N$, so AN is a graded S-co-r-submodule of M. The rest is clear. □

Corollary 2.9. *Let $S \subseteq h(R)$ be a multiplicatively closed subset of R and M be a graded R-module M. If $a_g \in h(R) \setminus Ann(M)$, then $a_g M$ is a graded S-co-r-submodule.*

Proof. It follows from Proposition 2.8. □

Proposition 2.10. *Let N be a non-zero graded submodule of a graded R-module M, and $S \subseteq h(R)$ be a multiplicatively closed subset of R such that $W_R^{gr}(M) \cap S = \emptyset$. Then the following are equivalent:*

(i) *N is a graded S-co-r-submodule of M.*

(ii) *There exists $s \in S$ such that $sN \subseteq a_g N$ and $sa_g \in h(R) \setminus W_R^{gr}(M)$.*

(iii) *There exists $s \in S$ such that $s(N :_M a_g) \subseteq N + (0 :_M a_g)$ and $sa_g \in h(R) \setminus W_R^{gr}(M)$.*

Proof. $(i) \Rightarrow (ii)$ Let N be a graded S-co-r-submodule of M. Since $a_gN \subseteq a_gN$ and $sa_gM = M$, it follows that $sN \subseteq a_gN$.

$(ii) \Rightarrow (i)$ Let $a_gN \subseteq K$ and $sa_gM = M$ for some $a_g \in h(R)$ and a graded submodule K of M. Then, by condition (ii), $sN \subseteq a_gN \subseteq K$ and hence N is a graded S-co-r-submodule.

$(ii) \Rightarrow (iii)$ Let $sm \in s(N :_M a_g)$ for some $m \in (N :_M a)$. Then we can write $m = \sum_{h\in G} m_h$. For each $h \in G$, we have $sa_gm_h \in sN \subseteq a_gN$, and hence $sa_gm_h = a_gn_{h'}$ for some $n_{h'} \in N \cap h(M)$. Therefore, $sm_h - n_{h'} \in (0 :_M a_g)$ and $sm_h = n_{h'} + sm_h - n_{h'} \in N + (0 :_M a_g)$. Consequently, $s(N :_M a_g) \subseteq N + (0 :_M a_g)$.

$(iii) \Rightarrow (ii)$ Let $y \in sN$. Then we can write $y = \sum_{h\in G} y_h$, so $y_h = sx_{h'}$ for some $x_{h'} \in N$. Since $sa_gM = M$, so $x_{h'} = sa_gm_{h''}$ for some $m_{h''} \in M \cap h(M)$. Hence, $sm_{h''} \in (N :_M a_g)$ and so $s^2m_{h''} \in s(N :_M a_g) \subseteq N + (0 :_M a_g)$. Then $s^2m_{h''} = n_t + m'_{t'}$ for some $n_t \in N \cap h(M)$ and $m'_{t'} \in (0 :_M a_g)$. Thus $sx_{h'} = s^2a_gm_{h''} = a_gn_t + a_gm'_{t'} = a_gn_t \in a_gN$. Therefore, $y_h \in a_gN$ and consequently $sN \subseteq a_gN$, as desired. □

A graded submodule N of a graded R-module M is called graded S-copure if there exists an element $s \in S$ such that $s(N :_M I) \subseteq N + (0 :_M I)$ for every graded ideal I of R.

By Proposition 2.10, we have the following corollary:

Corollary 2.11. *Every graded S-copure submodule of a graded R-module M is a graded S-co-r-submodule.*

The converse of Corollary 2.11 is not true in general. See the following example.

Example 2.12. Let $G = \mathbb{Z}_2$. Then consider the G-graded ring $\mathbb{Z}$, G-graded module $M = \mathbb{Z}_8 \times \mathbb{Z}_8$ with components $M_0 = \mathbb{Z}_8 \times \{0\}$ and $M_1 = \{0\} \times \mathbb{Z}_8$ and the multiplicatively closed subset $S = \{3^n | n \in \mathbb{N} \cup \{0\}\}$ of $\mathbb{Z}$. Then $\langle(\bar{2}, \bar{0})\rangle$ is a graded S-co-r-submodule of M. But $\langle(\bar{2}, \bar{0})\rangle$ is not a graded S-copure submodule of M.

Definition 2.13. Let $S \subseteq h(R)$ be a multiplicatively closed subset of R and let M be a graded R-module. We say that a graded S-co-r-submodule N of M is a minimal graded S-co-r-submodule of M, if there does not exist a graded S-co-r-submodule K of M such that $K \subseteq N$.

A non-zero graded submodule N of a graded R-module M with $S \cap Ann(N) = \emptyset$ is said to be graded S-second, if there exists $s \in S$ such that for each $a_g \in h(R)$, $sa_gN = sN$ or $sa_gN = 0$.

Proposition 2.14. *Let M be a graded R-module and $S \subseteq h(R)$ be a multiplicatively closed subset of R and N be a graded submodules of M with $S \cap Ann(N) = \emptyset$. Then N is a graded S-second submodule of M if and only if there exists $s \in S$ and whenever $a_gN \subseteq K$, where $a_g \in h(R)$ and K is a graded submodule of M, implies that either $sa_gN = 0$ or $sN \subseteq K$.*

Proof. The proof is clear. □

Proposition 2.15. *If N is a minimal graded S-co-r-submodule of a graded R-module M, then N is a graded S-second submodule.*

Proof. Let N be a minimal graded S-co-r-submodule with S-element $s \in S$. Suppose that $a_gN \subseteq K$ and $sN \nsubseteq K$, we show that $sa_g \in Ann(N)$. Let $sa_g \notin Ann(N)$. Then by

Proposition 2.8, a_gN is a graded S-co-r-submodule of M. Since N is a minimal graded S-co-r-submodule, we have $a_gN = N \subseteq K$ and so $sN \subseteq K$ which is a contradiction. Therefore, we get $sa_g \in Ann(N)$, as required. □

Let R_1 and R_2 be two G-graded rings. Then $R = R_1 \times R_2$ becomes a G-graded ring with homogeneous elements $h(R) = \bigcup_{g\in G} R_g$, where $R_g = (R_1)_g \times (R_2)_g$ for all $g \in G$. Let M_1 be a graded R_1-module and M_2 be a graded R_2-module. Then $M = M_1 \times M_2$ is a graded $R = R_1 \times R_2$-module with homogeneous elements $h(M) = \bigcup_{g\in G} M_g$, where $M_g = (M_1)_g \times (M_2)_g$ for all $g \in G$. Also, if $S_i \subseteq h(R_i)$ is a multiplicatively closed subset of R_i for each $i = 1, 2$, then $S = S_1 \times S_2 \subseteq h(R)$ is a multiplicatively closed subset of R. Furthermore, each graded submodule of M is of the form $N = N_1 \times N_2$ where N_i is a graded submodule of M_i.

Theorem 2.16. *Let $M = M_1 \times M_2$ and $R = R_1 \times R_2$ where M_i is a graded R_i-module for $i = 1, 2$. Then if $S = S_1 \times S_2 \subseteq h(R)$ is a multiplicatively closed subset of R and $N = N_1 \times N_2$ is a graded submodule of M, then the following statements are equivalent:*

- (i) *N is a graded S-co-r-submodule of M.*
- (ii) *$N_1 = 0$ and N_2 is a graded S_2-co-r-submodule of M_2 or N_1 is a graded S_1-co-r-submodule of M_1 and $N_2 = 0$ or N_1 is a graded S_1-co-r-submodule and N_2 is a graded S_2-co-r-submodules of M_2.*

Proof. $(i) \Rightarrow (ii)$ Suppose that N is a graded S-co-r-submodule of M with S-element $s = (s_1, s_2)$ and $N_1 = 0$. Then $N_2 \neq 0$ because N is a graded S-co-r-submodule of M. Let $(r_2)_gN_2 \subseteq K_2$ and $s_2(r_2)_g \notin W^{gr}_{R_2}(M_2)$ for some $(r_2)_g \in h(R_2)$. Thus $(0, (r_2)_g)(N_1 \times N_2) \subseteq M_1 \times K_2$ and $(s_1, s_2)(0, (r_2)_g) \notin W_R(M)$. Hence $(s_1, s_2)(N_1 \times N_2) \subseteq M_1 \times K_2$, so $s_2N_2 \subseteq K_2$, as needed. In other cases, a similar argument shows that $(i) \Rightarrow (ii)$.

$(ii) \Rightarrow (i)$ Let N_1 be a graded S_1-co-r-submodule of M_1 with S_1-element $s_1 \in S_1$ and N_2 be graded S_2-co-r-submodule of M_2 with S_2-element $s_2 \in S_2$. Assume that $((r_1)_g, (r_2)_g)(N_1 \times N_2) \subseteq K_1 \times K_2$ and $(s_1, s_2)((r_1)_g, (r_2)_g) \notin W^{gr}_R(M)$ where $((r_1)_g, (r_2)_g) \in h(R)$. Then $(r_1)_gN_1 \subseteq K_1$ and $s_1(r_1)_g \notin W^{gr}_{R_1}(M_1)$ and also, $(r_2)_gN_2 \subseteq K_2$ and $s_2(r_2)_g \notin W^{gr}_{R_2}(M_2)$. Since N_1 is a graded S_1-co-r-submodule of M_1 and N_2 is a graded S_2-co-r-submodule of M_2, we conclude $s_1N_1 \subseteq K_1$ and $s_2N_2 \subseteq K_2$. Hence $(s_1, s_2)(N_1 \times N_2) \subseteq K_1 \times K_2$. Therefore, $N = N_1 \times N_2$ is a graded S-co-r-submodule of M. □

Corollary 2.17. *Let $M = M_1 \times M_2 \times \ldots M_t$ and $R = R_1 \times R_2 \times \ldots R_t$ where M_i is a graded R_i-module for $i = 1, 2, \ldots, t$. Then if $S = S_1 \times S_2 \times \ldots S_t$ is a multiplicatively closed subset of R and $N = N_1 \times N_2 \times \ldots N_t$ is a graded submodule of M, then the following statements are equivalent:*

- (i) *N is a graded S-co-r-submodule of M.*
- (ii) *$N_i = 0$ for $i \in \{k_1, k_2, \ldots, k_m; m < t\} \subseteq \{1, 2, \ldots, t\}$ and N_i is a graded S_i-co-r-submodule of M_i for $i \in \{1, 2, \ldots, t\} \setminus \{k_1, k_2, \ldots, k_m; m < t\}$.*

Theorem 2.18. *Let $S \subseteq h(R)$ be a multiplicatively closed subset of R and N be a non-zero graded submodule of a graded R-module M. Then the following assertions hold:*

- (i) *N is a graded S-co-r-submodule of M with S-element $s \in S$ if and only if whenever I is a graded ideal of R such that $sI \cap (R \setminus W^{gr}_R(R)) \neq \emptyset$ and K is a graded submodule of M with $IN \subseteq K$, then $sN \subseteq K$.*

(ii) *If $Ann(N) \subseteq W_R^{gr}(M)$ and N is not a graded S-co-r-submodule of M, then there exists a graded ideal I of R such that $sI \cap (R \setminus W_R^{gr}(R)) \neq \emptyset$, $K \subset N$, $Ann(N) \subseteq I$ and $IN \subseteq K$.*

Proof. (i) Let N be a graded S-co-r-submodule of M with S-element $s \in S$. Let $IN \subseteq K$ for some graded ideal I of R with $sI \cap (h(R) \setminus W_R^{gr}(R)) \neq \emptyset$ and graded submodule K of M. There exists $a_g \in I \cap h(R)$ such that $sa_g \in I$ and $sa_gM = M$. Thus $sN \subseteq K$ since N is a graded S-co-r-submodule.

Conversely, let $a_gN \subseteq K$ and $sa_gM = M$ for some $a_g \in h(R)$ and graded submodule K of M. Take $I = \langle a_g \rangle$. Hence $sI \cap (h(R) \setminus W_R^{gr}(R)) \neq \emptyset$. Then by assumption, $sN \subseteq K$. Thus N is a graded S-co-r-submodule of M.

(ii) Since N is not a graded S-co-r-submodule of M, so for any $s \in S$ there exists $a_g \in h(R)$ and a graded submodule K of M such that $a_gN \subseteq K$ with $sa_gM \neq M$ and $sN \nsubseteq K$. Let $I = (K : N)$. Note that $a_g \in I$ and $sa_g \notin Ann(N)$ (since $sa_gM \neq M$). Thus $Ann(N) \subset I$. Now, we take $K = IN$. Therefore $K \subset N$ ($sN \nsubseteq K$), $sAnn(N) \subset I$ and $IN = (IN :_M I) \subseteq K$. □

Proposition 2.19. *Let M be a graded R-module, and let $S \subseteq h(R)$ be a multiplicatively closed subset of R. Let $K \subseteq N$ be graded submodules of M. If N/K is a graded S-co-r-submodule of R-module M/K and K is a graded co-r-submodule of M, then N is a graded S-co-r-submodule of M.*

Proof. Assume that N/K is a graded S-co-r-submodule of M/K with S-element $s \in S$. Let $a_gN \subseteq T$ and $sa_g \notin W_R^{gr}(M)$ for some $a_g \in h(R)$ and a graded submodule T of M. Since $a_gK \subseteq a_gN \subseteq T$ and $a_g \notin W_R^{gr}(M)$ ($sa_g \notin W_R^{gr}(M)$), it follows that $K \subseteq T$, because K is a graded co-r-submodule of M. Therefore, $a_g(N/K) = (a_gN + K)/K \subseteq T/K$ and clearly, $sa_g \notin W_R^{gr}(M/K)$. Hence $s(N/K) \subseteq T/K$. Consequently, $sN \subseteq T$ and so N is a graded S-co-r-submodule of M. □

Corollary 2.20. *Let $g : M \to M'$ be an epimorphism of graded R-modules, let $S \subseteq h(R)$ be a multiplicatively closed subset of R, and assume that $Ker(g)$ is a graded co-r-submodule of M. Then, if T is a graded S-co-r-submodule of M', it follows that $g^{-1}(T)$ is a graded S-co-r-submodule of M.*

Proof. It follows from Proposition 2.19. □

3. Graded S-co-n-submodules

Definition 3.1. (a) A non-zero graded submodule N of a graded R-module M is called a graded S-co-n-submodule, if $a_gN \subseteq K$ and $a_g \notin Grad(Ann(M))$ for $a_g \in h(R)$ and graded submodule K of M, then $N \subseteq K$.

(b) Let $S \subseteq h(R)$ be a multiplicatively closed subset of R and let M be a graded R-module. A non-zero graded submodule N of M with $Grad(Ann(M)) \cap S = \emptyset$ is called a graded S-co-n-submodule, if there exists $s \in S$ such that whenever $a_gN \subseteq K$ and $sa_g \notin Grad(Ann(M))$ for $a_g \in h(R)$ and graded submodule K of M, then $sN \subseteq K$.

Note that if $Grad(Ann(M)) \cap S \neq \emptyset$, then there exists $s \in S$ such that $s \in Grad(Ann(M))$. Consequently, for any $a_g \in h(R)$, we have $sa_g \in Grad(Ann(M))$. Indeed, in this case, every non-zero graded submodule is a graded S-co-n-submodule.

Remark 3.2. Let N be a non-zero graded submodule of a graded R-module M. If N is a graded co-n-submodule of M, then clearly, N is a graded S-co-n-submodule for any multiplicatively closed subset S of R. The following example shows that the converse is not true, in general. However, the classes of graded co-n-submodules and graded S-co-n-submodules coincide if $S \subseteq U^{gr}(R)$.

Example 3.3. Assume that $G = (\mathbb{Z}_2, +)$. Consider the G-graded ring $\mathbb{Z}$ and the graded $\mathbb{Z}$-module $M = \mathbb{Z} \times \mathbb{Z}_8$ with components $M_0 = \mathbb{Z} \times \{0\}$ and $M_1 = \{0\} \times \mathbb{Z}_8$ and the multiplicatively closed subset $S = \{2^n | n \in \mathbb{N} \cup \{0\}\}$ of $\mathbb{Z}$. Then the graded submodule $N = \langle (0, \bar{2}) \rangle$ is a graded S-co-n-submodule of M with S-element $s = 2$. But it is not a graded co-n-submodule, because $2N \subseteq \langle (0, \bar{4}) \rangle$, neither $2 \in Grad(Ann(M))$ nor $N \subseteq \langle (0, \bar{4}) \rangle$.

Proposition 3.4. *Let M be a graded R-module and $S \subseteq h(R)$ be a multiplicatively closed subset of R and N be a graded submodules of M with $S \cap Grad(Ann(N)) = \emptyset$. Then N is a graded S-secondary submodule of M if and only if there exists $s \in S$ and whenever $a_g N \subseteq K$, where $a_g \in h(R)$ and K is a graded submodule of M, implies that either $(sa_g)^n N = 0$ for some $n \in \mathbb{N}$ or $sN \subseteq K$.*

Proof. The proof is straightforward. □

By Proposition 3.4, we have if N is a graded S-co-n-submodule, then N is a graded S-secondary submodule, because $Grad(Ann(M)) \subseteq Grad(Ann(N))$. While, the following example shows that the concepts of graded S-second submodules and graded S-co-n-submodules are different in general.

Example 3.5. Assume that $G = (\mathbb{Z}_2, +)$. Consider the G-graded ring $\mathbb{Z}$ and the graded $\mathbb{Z}$-module $M = \mathbb{Z}_{12} \times \mathbb{Z}_{12}$ with components $M_0 = \mathbb{Z}_{12} \times \{0\}$ and $M_1 = \{0\} \times \mathbb{Z}_{12}$ and the multiplicatively closed subset $S = \{5^n | n \in \mathbb{N} \cup \{0\}\}$ of $\mathbb{Z}$. Then

(i) The graded submodule $N = \langle (\bar{3}, \bar{0}) \rangle$ is a graded S-co-n-submodule of $M = \mathbb{Z}_{12} \times \mathbb{Z}_{12}$. While N is not a graded S-second submodule of $M = \mathbb{Z}_{12} \times \mathbb{Z}_{12}$. Because, $3N \subseteq \langle (\bar{9}, \bar{0}) \rangle$, but for any $s \in S$, $(s \times 3)N \neq 0$ and $sN \nsubseteq \langle (\bar{9}, \bar{0}) \rangle$.

(ii) The submodule $N = \langle (\bar{0}, \bar{4}) \rangle$ is a graded S-second submodule of $M = \mathbb{Z}_{12} \times \mathbb{Z}_{12}$. While $N = \langle (\bar{0}, \bar{4}) \rangle$ is not a graded S-co-n-submodule of $M = \mathbb{Z}_{12} \times \mathbb{Z}_{12}$. Because, $3N = (\bar{0}, \bar{0})$, but for any $s \in S$, $sN \neq (\bar{0}, \bar{0})$ and $(s \times 3) \notin Grad(Ann(\mathbb{Z}_{12} \times \mathbb{Z}_{12}))$.

By Example 3.5 (ii), we conclude that a graded S-secondary submodule does not necessarily constitute a graded S-co-n-submodule.

Let R be a commutative graded ring, $S \subseteq h(R)$ be a multiplicatively closed subset of R and I be a graded ideal of R disjoint with S. We say that the graded ideal I is a graded S-n-ideal of R if there exists $s \in S$ such that for all $a_g, b_h \in h(R)$, $a_g b_h \in I$ and $sa_g \notin \mathfrak{n}^{gr}(R)$ ($\mathfrak{n}^{gr}(R)$ is the set of nilpotent inhomogeneous elements of R), then $sb_h \in I$.

Proposition 3.6. *Let N be a graded submodule of a graded R-module M. Then*

(i) *If N is a graded S-co-n-submodule of M such that $Grad(0) = Grad(Ann(M))$, then $Ann(N)$ is a graded S-n-ideal of R.*

(ii) *If M is a graded S-comultiplication R-module and $Ann(N)$ is a graded S-n-ideal of R, then N is a graded S-co-n-submodule of M.*

Proof. (i) Assume that N is a graded S-co-n-submodule of M with S-element s and let $a_g b_h \in Ann(N)$ and $sa_g \notin \mathfrak{n}^{gr}(R)$. We show that $sb_h \in Ann(N)$. Since $sa_g \notin \mathfrak{n}^{gr}(R)$, we conclude $sa_g \notin Grad(0)$ and so $sa_g \notin Grad(Ann(M))$ by assumption. As $a_g N \subseteq a_g N$ and $sa_g \notin Grad(Ann(M))$, we have $sN \subseteq aN$ because N is a graded S-co-n-submodule of M. Thus $sb_h N \subseteq a_g b_h N = 0$, hence $sb_h N = 0$ and $sb_h \in Ann(N)$.

(ii) Let $Ann(N)$ be a graded S-n-ideal of R with S-element $s \in S$. Assume that $a_g N \subseteq K$ with $sa_g \notin Grad(Ann(M))$ for some$a_g \in h(R)$ and a graded submodule K of M. Thus $a_g N Ann(K) \subseteq K Ann(K) = 0$, so $a_g N Ann(K) = 0$. Hence $a_g Ann(K) \subseteq Ann(N)$, since $sa_g \notin \mathfrak{n}^{gr}(R)$ and $Ann(N)$ is a graded S-n-ideal of R, we conclude $sAnn(K) \subseteq Ann(N)$. Therefore $(0 :_M Ann(N)) \subseteq (0 :_R sAnn(K))$. Since M is a graded S-comultiplication R-module, $sN \subseteq (0 :_M Ann(N)) \subseteq (0 :_R sAnn(K)) \subseteq sK \subseteq K$ and so $sN \subseteq K$, as needed. □

Theorem 3.7. *Let $\varphi : M_1 \to M_2$ be a monomorphism of graded R-modules, and let $S \subseteq h(R)$ be a multiplicatively closed subset of R. Then the following assertions hold:*

(i) *If N_1 is a graded S-co-n-submodule of M_1, then $\varphi(N_1)$ is a graded S-co-n-submodule of $\varphi(M_1)$.*

(ii) *If N_2 is a graded S-co-n-submodule of M_2 and $N_2 \subseteq \varphi(M_1)$, then $\varphi^{-1}(N_2)$ is a graded S-n-submodule of M_1.*

Proof. (i) Since φ is a monomorphism and $N_1 \neq 0$, we have $\varphi(N_1) \neq 0$. Let $a_g \in h(R)$ and K_2 be a graded submodule of M_2 such that $a_g \varphi(N_1) \subseteq K_2$. Then $a_g N_1 \subseteq \varphi^{-1}(K_2)$. Since N_1 is a graded S-co-n-submodule of M_1, there exists $s \in S$ such that $sN_1 \subseteq \varphi^{-1}(K_2)$ or $sa_g \in Grad(Ann(M))$. Hence, $s\varphi(N_1) \subseteq \varphi(\varphi^{-1}(K_2)) = \varphi(M) \cap K_2 \subseteq K_2$ or $sa_g \in Grad(Ann(\varphi(M_1)))$ (since φ is a monomorphism, so $Grad(Ann(M_1)) = Grad(Ann(\varphi(M_1)))$. Therefore, $\varphi(N_1)$ is a graded S-co-n-submodule of M_2.

(ii) It is clear that $\varphi^{-1}(N_2) \neq 0$ and let $a_g \in h(R)$ and K_1 be a graded submodule of M_1 such that $a_g \varphi^{-1}(N_2) \subseteq K_1$. Then $a_g N_2 = a_g(\varphi(M_1) \cap N_2) = a_g \varphi(\varphi^{-1}(N_2)) \subseteq \varphi(K_1)$. Since N_2 is a graded S-co-n-submodule of M_2, there exists $s \in S$ such that $sN_2 \subseteq \varphi(K_1)$ or $sa_g \in Grad(Ann(M_2))$. Hence, $\varphi^{-1}(sN_2) \subseteq \varphi^{-1}(\varphi(K_1)) = K_1$, and therefore $s\varphi^{-1}(N_2) \subseteq K_1$ or $sa_g \in Grad(Ann(M_1))$, as required. □

As an immediate consequence of the previous theorem, we have the following corollary.

Corollary 3.8. *Let M be a graded R-module and let $N \subseteq K$ be graded submodules of M. Then N is a graded S-co-n-submodule of K if and only if N is a graded S-co-n-submodule of M.*

Proposition 3.9. *Let M be a graded R-module. Then we have the following statements:*

(i) *If M is a graded S-secondary module, then M is a graded S-n-submodule of M.*

(ii) *The sum of an arbitrary non-empty set $\{N_i\}_{i=1}^{n}$ of graded S-co-n-submodules is a graded S-co-n-submodule of M.*

Proof. (i)The proof is straightforward.

(ii) Let $a_g \sum_{i=1}^{n} N_i \subseteq K$ for some $a_g \in h(R)$ and a graded submodule K of M. Then $a_g N_i \subseteq K$ for every i $(1 \leq i \leq n)$. Thus for each i, there exists $s_i \in S$ such that $s_i a_g \in Grad(Ann(M))$ or $s_i N_i \subseteq K$. Set $s = s_1 s_2 \ldots s_n$. Therefore, $sa_g \in Grad(Ann(M))$ or $sN \subseteq K$, as needed. □

Theorem 3.10. *Let M be a graded R-module, and let $S \subseteq h(R)$ be a multiplicatively closed subset of R. Then every proper graded submodule of M is a graded S-n-submodule of M if and only if every non-zero graded submodule of M is a graded S-co-n-submodule of M.*

Proof. ($\Rightarrow$) Let N be a non-zero graded submodule of M and $a_gN \subseteq K$ for some $a_g \in h(R)$ and a graded submodule K of M. Thus if $K = M$, then we are done. Let K be a proper graded submodule of M. Then K is a graded S-n-submodule of M. Assume that $m = \sum_{h\in G} m_h \in N$. Hence for each $h \in G$, $a_gm_h \in K$, so there exists $s \in S$ such that $sm_h \in K$ or $sa_g \in Grad(Ann(M))$. Therefore, $sN \subseteq K$ or $sa_g \in Grad(Ann(M))$, as required.

($\Leftarrow$) Let N be a proper graded submodule of M and $a_gm_h \in N$ for some $a_g \in h(R)$ and $m_h \in h(M)$. If $m_h = 0$, we are done. If $m_h \neq 0$, then $Rm_h \neq 0$ is a graded S-co-n-submodule. As $a_gRm_h \subseteq N$, there exists $s \in S$ such that $sa_g \in Grad(Ann(M))$ or $sRm_h \subseteq N$ ($sm_h \in N$). Thus N is a graded S-n-submodule of M. □

The following theorem establishes the correspondence between graded S-co-n-submodules and graded S-co-r-submodules.

Theorem 3.11. *Let M be a graded R-module, and let $S \subseteq h(R)$ be a multiplicatively closed subset of R. If N is a graded S-co-n-submodule of M, then N is also a graded S-co-r-submodule of M.*

Proof. Assume that N is a graded S-co-n-submodule of M with S-element $s \in S$. Let $a_gN \subseteq K$ with $sa_gM = M$. If $sa_g \in Grad(Ann(M))$, then there exists $n \in \mathbb{N}$ such that $(sa_g)^nM = 0$ and $(sa_g)^{n-1}M \neq 0$. However, since $sa_gM = M$, we would have $0 = (sa_g)^nM = (sa_g)^{n-1}M$, which is a contradiction. Therefore, $sa_g \notin Grad(Ann(M))$. Since N is a graded S-co-n-submodule, it follows that $sN \subseteq K$, as required. □

The following example demonstrates that the converse of the above theorem does not hold in general.

Example 3.12. Consider the $\mathbb{Z}_2$-graded ring $\mathbb{Z}$ and the graded $\mathbb{Z}$-module $\mathbb{Z}_{12} \times \{0\}$ and the multiplicatively closed subset $S = \{7^n | n \in \mathbb{N} \cup \{0\}\}$ of $\mathbb{Z}$. Then the graded submodule $N = \langle(\bar{3}, \bar{0})\rangle$ is a graded S-co-r-submodule, but it is not a graded S-co-n-submodule of $\mathbb{Z}_{12} \times \{0\}$. Because $2N \subseteq \langle(\bar{6}, \bar{0})\rangle$, but for any $s \in S$, neither $s \times 2 \in Grad(Ann(\mathbb{Z}_{12} \times \{0\}))$ nor $sN \subseteq \langle(\bar{3}, \bar{0})\rangle$.

4. Conclusions

In this paper, we introduced the notions of graded S-co-r-submodules and graded S-co-n-submodules of a graded module over a commutative graded ring. Several properties, examples, and characterizations of these graded submodules, particularly in graded S-multiplication modules and graded S-comultiplicaion modules, have been investigated. Moreover, we explored the behaviour of these graded submodules under graded module homomorphisms, quotient graded modules, and Cartesian product. Finally, we established the relationships between these concepts.

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