



Research Paper

SOME PROPERTIES OF EXTENDING MODULES AND THEIR GENERALIZATIONS

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ARTICLE INFO

Article history:

Received: 13 October 2025

Accepted: 4 July 2026

Communicated by Ahmad Yousefian Darani

Keywords:

CS-module*CESS*-module*CLS*-moduleweak *CS*-module

ABSTRACT

In this paper, we discuss about the direct sum of a *CS*-module with essential socle and a module with zero socle. Next, we discuss about the direct sum of two *CS*-modules and also two *CLS*-modules in some cases.

MSC:

16D10; 16D70; 16D99

1. INTRODUCTION

All rings R have identities and all modules are unitary right R -modules, unless otherwise stated. Let us give some basic notations and terminologies. If M is a module, then the notations $A \leq M$, $A \leq_e M$ mean A is a submodule of M , A is an essential submodule of M , respectively. The singular submodule of M is $Z(M) = \{m \in M \mid \text{ann}_r(m) \leq_e R_R\}$. M is called a singular module if $Z(M) = M$; and M is nonsingular if $Z(M) = 0$.

Let $S \leq M$. A submodule $C \leq M$ is said to be a complement to S (in M) if C is maximal with respect to the property that $C \cap S = 0$. A submodule C of M is a complement

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(submodule) if there exists a submodule $S \leq M$ such that C is a complement to S in M . Recall from [4] and [12] that a submodule N of a module M is said to be $\mathcal{S}$ -closed if M/N is nonsingular (In [11], Tercan and Yücel call $\mathcal{S}$ -closed by “ z -closed”). A submodule K of M is called closed (in M) if K has no proper essential extension in M . In general, closed submodules need not be $\mathcal{S}$ -closed. For example, 0 is a closed submodule of any module M , but 0 is $\mathcal{S}$ -closed in M only if M is nonsingular. It is well known that, every $\mathcal{S}$ -closed submodule of a module M is a closed submodule, and every closed submodule of a nonsingular module is $\mathcal{S}$ -closed in M . (For example, see [7, Lemma 2.3] or [4, Proposition 2.4].) Also, it is well known (and easy to prove) that a submodule K of M is closed if and only if K is complement (see, for example, [3, 1.10] or [5, Proposition 6.32]).

CESS-modules introduced by P. F. Smith [8], which is a generalization of *CS*-modules. Recall that a module M is called a *CS*-module (or extending module) if every complement submodule of M is a direct summand of M (equivalently, every submodule is essential in a direct summand of M), weak *CS*-module if every semisimple submodule is essential in a direct summand of M [8], a *CESS*-module if every complement submodule of M with essential socle is a direct summand of M , and a *CLS*-module if every $\mathcal{S}$ -closed submodule of M is a direct summand of M [10].

Let M be a *CESS*-module. Then by [8, corollary 1.6], $M = M_1 \oplus M_2$ where M_1 is a *CS*-module with essential socle and M_2 with zero socle, but the converse is false, in general, by [13, Question 1]. In this paper, we discuss about the direct sum of a *CS*-module with essential socle and a module with zero socle. Next, we discuss about the direct sum of two *CS*-modules and also two *CLS*-modules in some cases. Precisely, if $M = M_1 \oplus M_2$ is a *UC*-module for some *CS*-modules M_1 and M_2 , then M is a *CS*-module. Also, if $M = M_1 \oplus M_2$ is a Duo module for some *CLS*-modules M_1, M_2 , then M is a *CLS*-module.

2. DIRECT SUM OF A *CS*-MODULE WITH ESSENTIAL SOCLE AND A MODULE WITH ZERO SOCLE

Let M be a *CESS*-module. Then by [8, corollary 1.6], $M = M_1 \oplus M_2$ where M_1 is a *CS*-module with essential socle and M_2 with zero socle. In [8, Question 1.7] it is asked whether is true the converse? This is shown to be false, in general, by [13, Question 1]. In fact, let $R = \mathbb{Z} \oplus \mathbb{Z}_2(\infty)$ be the trivial extension of $\mathbb{Z}$ and the $\mathbb{Z}$ -module $\mathbb{Z}_2(\infty)$. Let $M_1 = R_R$ and $M_2 = R/I$ where $I = (0) \oplus \mathbb{Z}_2(\infty)$. Then M_1 is a *CS*-module with $\text{Soc}(M_1) = (0) \oplus \mathbb{Z}_2$ is essential in M_1 and M_2 has zero socle but $M = M_1 \oplus M_2$ is not a *CESS*-module. Note that $N = \{((2k, a), (k, 0) + I) : k \in \mathbb{Z} \text{ and } a \in \mathbb{Z}_2(\infty)\}$ is a complement submodule of M with essential socle which is not a direct summand of M .

The next result shows that weak *CS*-modules are better behaved.

Proposition 2.1. *Let M be a module such that $M = M_1 \oplus M_2$ for some *CS*-module M_1 and module M_2 with zero socle. Then M is weak *CS*-module.*

Proof. Clearly *CS*-modules are weak *CS*. Thus M_1 is a weak *CS*-module and so by [8, Theorem 1.12] $M_1 \oplus M_2$ is a weak *CS*-module, i.e. M is a weak *CS*-module. $\square$

By Zorn's Lemma, for every submodule N of a module M there exists a closed submodule K of M such that $N \leq_e K$, and in this case, K is called a closure of N (in M).

Lemma 2.2. *Let M be a module such that $M = M_1 \oplus M_2$ for some module M_1 with essential socle and module M_2 with zero socle. Consider the following statements:*

- (i) M_1 is semisimple;
- (ii) $\text{Soc } M$ has a unique closure;
- (iii) $\text{Hom}(N, M_2) = 0$ for every submodule N of M_1 ;
- (iv) M_2 is M_1 -injective;
- (v) M is a CESS-module.

Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv). Furthermore, if M_1 is a CS-module then (iv) $\Rightarrow$ (v).

Proof. See the proof of [2, Lemma 2.3].

Corollary 2.3. *Let $M = M_1 \oplus M_2$ be a nonsingular module for some CS-module M_1 with essential socle and module M_2 with zero socle. Then M is a CESS-module.*

Proof. Since M is nonsingular, it is an UC-module. Therefore, $\text{Soc } M$ has a unique closure. Now, as M_1 is a CS-module, so it follows from Lemma 2.2 that M is a CESS-module. $\square$

Lemma 2.4. *If N is a direct summand of M , then N is closed in M .*

Proof. We have $M = N \oplus K$ for some $K \leq M$. It follows from [5, example 6.17(1)] that N is a complement to K in M . Hence N is a complement submodule of M and so N is closed in M . $\square$

Corollary 2.5. *Let M be a module such that $M = M_1 \oplus M_2$ for some module M_1 with essential socle and module M_2 with zero socle. Then M_1 is a closure of $\text{Soc } M$.*

Proof. As M_2 has zero socle, we have $\text{Soc } M = \text{Soc } M_1$. But M_1 has essential socle, then $\text{Soc } M$ is essential in M_1 . Also since $M = M_1 \oplus M_2$ it follows from 2.4 that M_1 is closed in M . Therefore, M_1 is a closure of $\text{Soc } M$.

Proposition 2.6. *Let M be a module such that $M = M_1 \oplus M_2$ for some module M_1 with essential socle and nonsingular module M_2 with zero socle. Then M_1 is the only closure of $\text{Soc } M$, i.e. $\text{Soc } M$ is a UC-module.*

Proof. In view of Lemma 2.5, M_1 is a closure of $\text{Soc } M$. Now let K be a closure of $\text{Soc } M$. Note that $\text{Soc } M = \text{Soc } M_1$, hence $\text{Soc } M \subseteq M_1 \cap K \subseteq K$. But $\text{Soc } M \leq_e K$, so $M_1 \cap K \leq_e K$ and this implies that $K/(M_1 \cap K)$ is singular. On the other hand, $K/(M_1 \cap K) \cong (M_1 + K)/M_1 \leq M/M_1 \cong M_2$ and M_2 is nonsingular, so that $K/(M_1 \cap K)$ is nonsingular. It follows that $K \subseteq M_1$. Now since $\text{Soc } M_1 \subseteq K \subseteq M_1$ and M_1 has essential socle, we deduce that K is essential in M_1 . But K is closed in M , so $K = M_1$. $\square$

Corollary 2.7. *Let M be a module such that $M = M_1 \oplus M_2$ for some module M_1 with essential socle and nonsingular module M_2 with zero socle. Then $\text{Hom}(N, M_2) = 0$ for every submodule N of M_1 . In particular, M_2 is M_1 -injective.*

Proof. The result follows by Proposition 2.6 and Lemma 2.2. $\square$

Lemma 2.8. *Let M be a semi-artinian module and N be a module with zero socle. Then $\text{Hom}(M, N) = 0$. $\square$*

Proof. Let $\varphi \in \text{Hom}(M, N)$. Then $M/\text{Ker } \varphi \cong \text{Im } \varphi$. To prove that $\varphi = 0$ it is sufficient to show that $\text{Ker } \varphi = M$. If $\text{Ker } \varphi \neq M$ then $M/\text{Ker } \varphi \neq 0$ and since M is semi-artinian so $\text{Soc}(M/\text{Ker } \varphi) \neq 0$. Therefore, $0 \neq \text{Soc}(M/\text{Ker } \varphi) \simeq \text{Soc}(\text{Im } \varphi) \subseteq \text{Soc } N = 0$, which is a contradiction, as required. $\square$

Proposition 2.9. *Let $M = M_1 \oplus M_2$ for some semi-artinian CS -module M_1 and module M_2 with zero socle. Then M is a $CESS$ -module.*

Proof. Since M_1 is semi-artinian, so $\text{Soc } M_1$ is essential in M_1 . By Lemma 2.2, it is enough to show that $\text{Hom}(N, M_2) = 0$ for all $N \leq M_1$. Let $N \leq M_1$. Thus N is semi-artinian and so $\text{Hom}(N, M_2) = 0$ by Lemma 2.8, as required. $\square$

Corollary 2.10. *Let $M = M_1 \oplus M_2$ for some Artinian CS -module M_1 and module M_2 with zero socle. Then M is a $CESS$ -module.*

3. DECOMPOSITION TO A CS -MODULE WITH ESSENTIAL SOCLE AND A MODULE WITH ZERO SOCLE

Proposition 3.1. *Let M be a quasi-continuous module. Then $M = M_1 \oplus M_2$ for some CS -module M_1 with essential socle and module M_2 with zero socle such that M_1, M_2 are relatively injective.*

Proof. By the fact that every quasi-continuous module is a CS -module, so M is a CS -module, and thus M is a $CESS$ -module. It follows by [8, corollary 1.6] that $M = M_1 \oplus M_2$, where M_1 is a CS -module with essential socle and M_2 is a module with zero socle. Also since $M = M_1 \oplus M_2$ is quasi-continuous, so by [6, Proposition 2.10], M_1 and M_2 are relatively injective. $\square$

Corollary 3.2. *Let M be a quasi-injective module. Then $M = M_1 \oplus M_2$ for some CS -module M_1 with essential socle and module M_2 with zero socle such that M_i is M_j -injective for $i, j = 1, 2$.*

Proof. Inasmuch as every quasi-injective module is a quasi-continuous module, it follows from Proposition 3.1 that $M = M_1 \oplus M_2$, where M_1 is a CS -module with essential socle and M_2 is a module with zero socle. Also since $M = M_1 \oplus M_2$ is quasi-injective so by [6, Proposition 1.17], M_i is M_j -injective for $i, j = 1, 2$. $\square$

4. THE DIRECT SUM OF SOME MODULES

It is well-known that a direct sum of CS -modules need not be CS . For example, if p is a prime number then $\mathbb{Z}/\mathbb{Z}p$ and $\mathbb{Z}/\mathbb{Z}p^3$ are both CS -modules but $(\mathbb{Z}/\mathbb{Z}p) \oplus (\mathbb{Z}/\mathbb{Z}p^3)$ is not CS . So it is natural to ask when the direct sum of CS -modules is a CS -module. The next result shows that if direct sum of two CS -modules is a UC -module then the direct sum is a CS -module.

Proposition 4.1. *Let $M = M_1 \oplus M_2$ be a distributive module for some CS -modules M_1 and M_2 . Then M is a CS -module.*

Proof. Let N be a complement submodule of M . By the fact that every distributive module is a UC -module, it follows from [9, Theorem] that $N \cap M_1$ is a complement submodule of M_1 . But M_1 is a CS -module, so $N \cap M_1$ is a direct summand of M_1 . This implies that

$M_1 = (N \cap M_1) \oplus M'_1$ for some $M'_1 \leq M_1$. Similarly, $M_2 = (N \cap M_2) \oplus M'_2$ for some $M'_2 \leq M_2$. Therefore, since M is a distributive module, we can write $M = (N \cap M_1) \oplus (N \cap M_2) \oplus (M'_1 \oplus M'_2) = N \oplus M'_1 \oplus M'_2$, and thus N is a direct summand of M . Consequently, M is a CS -module. $\square$

A direct sum of CLS -modules need not be CLS . For example, $\mathbb{Z}$ and $\mathbb{Z}_{(2)}$ are both CLS -modules where $\mathbb{Z}_{(2)} = \{a/b | a, b \in \mathbb{Z}, b \text{ is odd}\}$, but $\mathbb{Z} \oplus \mathbb{Z}_{(2)}$ is not CLS (see [10, p. 1560]). The next result shows that if direct sum of two CLS -modules is a Duo module then the direct sum is a CLS -module.

Proposition 4.2. *Let $M = M_1 \oplus M_2$ be a Duo module for some CLS -modules M_1, M_2 . Then M is a CLS -module.*

Proof. Let N be an $\mathcal{S}$ -closed submodule of M . We have $M_1/(M_1 \cap N) \cong (M_1 + N)/N \leq M/N$. This implies that $M_1 \cap N$ is $\mathcal{S}$ -closed in M_1 and hence $M_1 \cap N$ is a direct summand of M_1 . Hence $M_1 = (M_1 \cap N) \oplus M'_1$ for some $M'_1 \leq M_1$. Similarly $M_2 = (M_2 \cap N) \oplus M'_2$ for some $M'_2 \leq M_2$. Therefore $M = (M_1 \cap N) \oplus (M_2 \cap N) \oplus (M'_1 \oplus M'_2)$. But M is a Duo module, and hence N is a fully invariant submodule of M and so we can write $N = (M_1 \cap N) \oplus (M_2 \cap N)$. Then $M = N \oplus (M'_1 \oplus M'_2)$. Thus N is a direct summand of M . It follows that M is a CLS -module. $\square$

The following result shows that the direct sum of a finitely generated module and a cyclic singular module over a Dedekind domain is a weak CS -module.

Proposition 4.3. *Let R be a Dedekind domain and $M = M_1 \oplus M_2$ for some finitely generated module M_1 and cyclic singular module M_2 . Then M is a weak CS -module.*

Proof. By [8, Theorem 1.16], M_1 is a weak CS -module. On the other hand, since M_2 is cyclic so $M_2 = Rx \cong R/\text{ann } x$ for some $x \in M_2$. But M_2 is singular so $x \in Z(M_2)$ and thus $\text{ann } x \leq_e R$. Since R has SP , then by [1, Theorem 1], $R/\text{ann } x$ is semisimple. Hence we have proved that M_2 is semisimple. Therefore, by [8, Theorem 1.10] M is a weak CS -module. $\square$

5. OTHER RESULTS

Proposition 5.1. *Let M be an R -module and $N \leq M$.*

- (a) *If M is nonsingular and N is a direct summand of M , then N is $\mathcal{S}$ -closed in M ;*
- (b) *If N is injective or M is semisimple, then N is closed in M ;*
- (c) *Assume that M is nonsingular. If N is injective or M is semisimple, then N is $\mathcal{S}$ -closed in M ;*
- (d) *If R is semisimple, then N is $\mathcal{S}$ -closed in M .*

Proof. (a) N is direct summand of M . Hence $M = N \oplus K$ for some $K \leq M$. Then $Z(M/N) \cong Z(K) \leq Z(M)$. But since M is nonsingular, so $Z(M) = 0$ and it follows $Z(M/N) = 0$, i.e N is $\mathcal{S}$ -closed in M .

(b) It is clear by 2.4.

(c) N is closed by (b). But since M is nonsingular, so by [4, Proposition 3.4] N is $\mathcal{S}$ -closed.

(d) If R is semisimple, then M is semisimple and nonsingular. Now the result is followed by (c). $\square$

Proposition 5.2. *Let M be a semisimple module and $N \leq M$.*

- (a) *If N is $\mathcal{S}$ -closed in M , then M/N is projective;*
- (b) *If M is nonsingular, then N is projective.*

Proof. (a) Inasmuch as every homomorphic image of a semisimple module is semisimple, so (M/N) is semisimple. But M/N is nonsingular, hence according to [4, corollary 1.25], M/N is projective.

(b) Clearly N is semisimple and nonsingular. Hence again by [4, corollary 1.25], N is projective. $\square$

Proposition 5.3. *Let N be a closed submodule of M with $\text{Soc } N \leq_e N$ and N is an intersection of essential submodules of M . Then N is an essential closure of $\text{Soc } M$.*

Proof. $\text{Soc } M$ is the intersection of all essential submodules of M . Hence by hypothesis, $\text{Soc } M \subseteq N$. Inasmuch as N has essential socle, it follows from $\text{Soc } N \subseteq \text{Soc } M \subseteq N$ that $\text{Soc } M \leq_e N$. But N is closed so N is an essential closure of $\text{Soc } M$. $\square$

Proposition 5.4. *Let R be a commutative domain, $f : M \rightarrow N$ an R -monomorphism. If N is torsion-free then M is torsion-free if and only if M is a UC -module.*

Proof. Since every nonsingular module is a UC -module so the necessity is clear. Conversely, suppose that M is a UC -module. If $M = 0$ the result is trivial, so let $M \neq 0$. By [2, Theorem 3.2] M is torsion or torsion-free. If M is torsion, then $f(M) = f(T(M)) \subseteq T(N) = 0$. Hence $f(M) = 0$. But since f is a monomorphism, it follows that $M = 0$ which is a contradiction. Hence M is torsion-free. $\square$

AUTHOR CONTRIBUTIONS STATEMENT

All authors wrote and reviewed the manuscript.

DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

Acknowledgments The authors are deeply grateful to the referee for careful reading of the original manuscript and valuable suggestions.

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