DOI: 10.22098/jhs.2026.18443.1138

Journal of Hyperstructures, 15 (2) (2026), 249-260

Journal Homepage: https://jhs.uma.ac.ir/

Research Paper

ON FUZZY GROUPOID-POSET STRUCTURES

KARIM GHADIMI[1,*], AKBAR REZAEI[2] AND HEE SIK KIM[3]

[1]Department of Mathematics, Payame Noor University, Tehran, Iran, kghadimi@pnu.ac.ir.

[2]Department of Mathematics, Payame Noor University, Tehran, Iran, rezaei@pnu.ac.ir

[3]Department of Mathematics, Hanyang University, Seoul, Korea, heekim@hanyang.ac.kr

ARTICLE INFO

Article history:
Received: 27 September 2025
Accepted: 11 May 2026
Communicated by Yuming Feng

Keywords:
groupoid
fuzzy partially ordered groupoid
d/BCK-algebra

MSC:
20N02; 06F35

ABSTRACT

In this paper, we consider a groupoid $(X, *)$ equipped with a poset structure $(X, \leq)$ and study fuzzy partially ordered groupoids where induced by a fuzzy subset μ. We define a set $P_\mu(*; \leq)$ and explore its properties. Specifically, we demonstrate that if $(X, *, 0)$ is a left zero semigroup or an idempotent groupoid, then $(X, *)$ is fuzzy fully dominated by a poset $(X, \leq)$ induced by a fuzzy subset μ.

1. Introduction

Groupoid theory has garnered significant attention in the literature, with researchers exploring its connections to algebraic structures (e.g., [3, 4, 17, 21]). It has combined with the theory of general algebraic structures. Several properties of the groupoid theory have been developed with BCK-algebras and its related topics [7, 12, 14]. Neggers and Kim later introduced the concept of d-algebras [18], a useful generalization of BCK-algebras, and explored numerous relationships between d-algebras and BCK-algebras, showing that d-algebras are among the least associative algebras. Ahn and Kim [1] developed implicative and commutative d-algebras that are not BCK-algebras, proving that these are indeed generalizations

*Address correspondence to K. Ghadimi; Department of Mathematics, Payame Noor University, P. O. Box 19395-4697, Tehran, Iran, E-mail: kghadimi@pnu.ac.ir.

This work is licensed under a Creative Commons Attribution-NonCommercial 4.0 International License.
Copyright © 2026 The Author(s). Published by University of Mohaghegh Ardabili.

of BCK-algebra concepts. Meanwhile, the study of BCK-algebras and related structures has revealed rich order-theoretic properties [6] and [8], yet many questions remain regarding how arbitrary groupoids interact with posets.

Zadeh [23] introduced the notion of a fuzzy subset as a function from a set into interval, and Rosenfeld [22] applied this concept to the theory of groupoid and groups. Mordeson and Malik [16] published a book, Fuzzy commutative algebra, which are fuzzifications of several classical algebras, and Ahsan et al. [2] published a book, Fuzzy semirings with applications to automata theory. Kim and Neggers [11] applied it to pogroupoids which are algebraic representations of partially ordered sets, and obtained an equivalent condition for some relation to be transitive for any fuzzy subset. Han et al. [5] discussed on linear fuzzifications of groupoids with special emphasis on BCK-algebras. Liu et al. [13] studied the notion of hyperfuzzy groupoids as a natural extension of the basic concepts of fuzzy groupoids.

In this paper, we study a groupoid $(X, *)$ equipped with a poset structure $(X, \leq)$ and fuzzy partially ordered groupoids where induced by a fuzzy subset μ. We define a set $P_\mu(*; \leq)$ and explore its properties. Specifically, we prove that if $(X, *)$ is a left zero semigroup or an idempotent groupoid, then $X = P_\mu(*; \leq)$ as a set, establishing a significant connection between the groupoid equipped with a poset and fuzzy partially ordered groupoid where induced by a fuzzy subset μ i.e., is a connecting. Furthermore, if $(X, *)$ is a left zero semigroup or an idempotent groupoid, then $(X, *)$ is fuzzy fully dominated by a poset $(X, \leq)$ induced by a fuzzy subset μ.

2. Preliminaries

A groupoid $(X, *)$ is said to be *right* $\big($*left*, resp.$\big)$ *compatible with a poset* $(X, \leq)$ if $x \leq y$ implies $x * z \leq y * z$ $\big(z * x \leq z * y$, resp.$\big)$ for all $z \in X$. A groupoid $(X, *)$ is said to be *compatible with a poset* $(X, \leq)$ if it is both right compatible and left compatible with a poset $(X, \leq)$. Note that it is equivalent to that $x \leq y, u \leq w$ implies $x * u \leq y * w$. A groupoid $(X, *)$ is said to be *medial* if $(x * y) * (r * s) = (x * r) * (y * s)$ for all $x, y, r, s \in X$.

A *BCK-algebra* [6, 8, 9, 15] is a non-empty set X with a constant 0 and a binary operation "$*$" satisfying the following axioms:

(I) $x * x = 0$,
(II) $0 * x = 0$,
(III) $x * y = 0$ and $y * x = 0$ imply $x = y$,
(IV) $\big(x * (x * y)\big) * y = 0$,
(V) $\big((x * y) * (x * z)\big) * (z * y) = 0$,

for all $x, y, z \in X$.

Theorem 2.1. ([15]) *Let $(X, *, 0)$ be a BCK-algebra. Then for any x, y and z in X, the following hold:*

(i) $x * y \leq z$ *implies* $x * z \leq y$,
(ii) $(x * z) * (y * z) \leq x * y$,
(iii) $x \leq y$ *implies* $x * z \leq y * z$,
(iv) $x * y \leq x$,
(v) $x * 0 = x$.

An algebra $(X, *, 0)$ of type $(2, 0)$ is called a *d-algebra* ([18]) if it satisfies (I), (II) and (III). A d/BCK-algebra $(X, *, 0)$ is said to be positive implicative if $(x * z) * (y * z) = (x * y) * z$ for all $x, y, z \in X$. An algebra $(X, *, \leq)$ is said to be a *partially ordered groupoid* ([20]) if

- $(X, *)$ is a groupoid,
- $(X, \leq)$ is a partially ordered set.

Let $(X, *)$ be a groupoid. A non-empty subset I of X is said to be a

- *right* $\big($*left,* resp.$\big)$ *ideal*: $x \in I, y \in X$ implies $x * y \in I$, i.e., $I * X \subseteq I$ $\big(X * I \subseteq I$, resp.$\big)$,
- *right* $\big($*left,* resp.$\big)$ *semi-ideal*: $x*y \in I, z \in X$ implies $(x*z)*(y*z) \in I$ $\big((z*x)*(z*y) \in I$, resp.$\big)$,
- *semi-ideal*: $x * y \in I, z \in X$ implies $(x * z) * (y * z) \in I$ and $(z * x) * (z * y) \in I$,
- *reverse right* $\big($*left,* resp.$\big)$ *semi-ideal*: $(x * z) * (y * z) \in I$ $\big((z * x) * (z * y) \in I$, resp.$\big)$, $z \in X$ implies $x * y \in I$,
- *reverse semi-ideal*: $(x * z) * (y * z) \in I$ and $(z * x) * (z * y) \in I$, $z \in X$ imply $x * y \in I$

for all $x, y \in X$ ([20]).

Example 2.2. ([20]) (i) Let X be the set of all real numbers. Define a binary operation "$*$" on X by $x*y := x(x-y)$ for all $x, y \in X$. Then $(X, *, 0)$ is a d-algebra. If we let $I := \{x*0 \,|\, x \in X\}$, then $I = \{x(x-0) \,|\, x \in X\} = \{x^2 \,|\, x \in X\}$, and hence $\{0\} \neq I \neq X$. We claim that I is a right semi-ideal of X. Indeed, if $x * 0 \in I$ and $z \in X$, then $(x * z) * (0 * z) = (x * z) * 0 \in I$.

(ii) Let $X := \mathbf{R}$ be the set of all real numbers and let "$-$" be the usual subtraction in $\mathbf{R}$. Then $(X, -)$ is a reverse semi-ideal of itself, since $x - y = (x - z) - (y - z)$ for all $x, y \in X$.

3. Fuzzy partially ordered groupoids

Let $(X, *, \leq)$ be a partially ordered groupoid. A map $\mu : X \to [0, 1]$ is said to be a *fuzzy subgroupoid* of $(X, *)$ if, for all $x, y \in X$,

$$\mu(x * y) \geq \min\{\mu(x), \mu(y)\}.$$

A map $\mu : X \to [0, 1]$ is said to be a *order preserving mapping* if

$$x \leq y \;\; \Rightarrow \;\; \mu(x) \leq \mu(y).$$

A map $\mu : X \to [0, 1]$ is said to be a *fuzzy partially ordered subgroupoid* if it is both a fuzzy subgroupoid of $(X, *)$ and a order preserving mapping. A map $\mu : X \to [0, 1]$ is said to be *full* if $\mu(x) \leq \mu(y)$ implies $x \leq y$. A map $\mu : X \to [0, 1]$ is said to be a *fuzzy right* $\big($*left, resp.*$\big)$ *compatible* if $\mu(x) \leq \mu(y)$ implies $\mu(x * z) \leq \mu(y * z)$ $\big(\mu(z * x) \leq \mu(z * y)$, resp.$\big)$ for all $z \in X$.

Proposition 3.1. *If a fuzzy partially ordered subgroupoid $\mu : (X, *) \to [0, 1]$ is both full and fuzzy right* $\big($*left, resp.*$\big)$ *compatible over a poset $(X, \leq)$, then $(X, \leq)$ is right* $\big($*left, resp.*$\big)$ *compatible over a groupoid $(X, *)$.*

Proof. Assume that $x \leq y$. Since μ is order preserving, we have $\mu(x) \leq \mu(y)$. Since μ is fuzzy right $\big($left, resp.$\big)$ compatible, we obtain $\mu(x * z) \leq \mu(y * z)$ $\big(\mu(z * x) \leq \mu(z * y)$, resp.$\big)$ for all $z \in X$. It follows from μ is full that $x * z \leq y * z$ $\big(z * x \leq z * y$, resp.$\big)$, proving the proposition. □

Theorem 3.2. *Let $\mu : (X, *) \to [0, 1]$ be a full. If $(X, *)$ is right* $\big($*left, resp.*$\big)$ *compatible over a poset $(X, \leq)$, then μ is fuzzy right* $\big($*left, resp.*$\big)$ *compatible.*

Proof. Assume $\mu(x) \leq \mu(y)$. Since μ is full, we have $x \leq y$. Since $(X, *)$ is right $\big($left, resp.$\big)$ compatible over a poset $(X, \leq)$, we obtain $x * z \leq y * z$ $\big(z * x \leq z * y$, resp.$\big)$ for all $z \in X$. Since μ is order preserving mapping, we have $\mu(x * z) \leq \mu(y * z)$ $\big(\mu(z * x) \leq \mu(z * y)$, resp.$\big)$ for all $z \in X$, proving the proposition. □

A fuzzy subgroupoid $\mu : (X, *) \to [0, 1]$ is said to be a

- *fuzzy right* $\big($*left, resp.*$\big)$ *semi-ideal* if, for all $x, y, z \in X$, $\mu\big((x * z) * (y * z)\big) \leq \mu(x * y)$ $\big(\mu\big((z * x) * (z * y)\big) \leq \mu(x * y)$, resp.$\big)$.
- *fuzzy semi-ideal* if, for all $x, y, z \in X$, $\mu\big((x * z) * (y * z)\big) \leq \mu(x * y)$ and $\mu\big((z * x) * (z * y)\big) \leq \mu(x * y)$.
- *fuzzy reverse right semi-ideal* $\big($*fuzzy reverse left semi-ideal*, resp.$\big)$ if, for all $x, y, z \in X$, $\mu\big((x * z) * (y * z)\big) \geq \mu(x * y)$ $\big(\mu\big((z * x) * (z * y)\big) \geq \mu(x * y)$, resp.$\big)$.
- *fuzzy reverse semi-ideal* if, for all $x, y, z \in X$, $\mu\big((x * z) * (y * z)\big) \geq \mu(x * y)$ and $\mu\big((z * x) * (z * y)\big) \geq \mu(x * y)$.

Example 3.3. Let $X := \mathbf{R}$ be the set of all real numbers and let "$-$" be the usual subtraction on X. Since $(x - z) - (y - z) = x - y$ for all $x, y, z \in X$, every fuzzy subgroupoid of $(X, -)$ is a fuzzy right semi-ideal. Moreover, if a fuzzy subgroupoid μ of $(X, -)$ satisfies the condition, $\mu(x - y) = \mu(y - x)$ for all $x, y \in X$, then it is a fuzzy left semi-ideal of $(X, -)$.

Theorem 3.4. *Let $(X, *, 0)$ be a BCK-algebra and $\mu : (X, *) \to [0, 1]$ be a fuzzy partially ordered subgroupoid. The following hold:*

(i) *μ is a fuzzy right semi-ideal,*
(ii) *if $\mu(x * y) = \mu(y * x)$ for all $x, y \in X$, then μ is a fuzzy semi-ideal.*

Proof. (i) Assume $(X, *, 0)$ is a BCK-algebra. By Theorem 2.1(ii)

$$(x * z) * (y * z) \leq x * y$$

for all $x, y, z \in X$. Since μ is order preserving mapping, we obtain $\mu\big((x*z)*(y*z)\big) \leq \mu(x*y)$. Hence μ is a fuzzy right semi-ideal.

(ii) Assume $(X, *, 0)$ is a BCK-algebra. Then

$$(z * x) * (z * y) \leq y * x.$$

for all $x, y, z \in X$. Since μ is order preserving mapping, we obtain $\mu\big((z * x) * (z * y)\big) \leq \mu(y * x) = \mu(x * y)$. Hence μ is a fuzzy semi-ideal. □

Corollary 3.5. *Let $(X, *, 0)$ be a BCK-algebra. If $\mu : (X, *) \to [0, 1]$ is fuzzy partially ordered subgroupoid and $\mu(x * y) = \mu(y * x)$ for all $x, y \in X$, then μ is a fuzzy semi-ideal.*

Proof. We conclude with Theorem 3.4. □

Theorem 3.6. *Let $(X, *, 0)$ be a BCK-algebra and $\mu : (X, *) \to [0, 1]$ be a fuzzy set. The following hold:*

(i) *if μ is a fuzzy right semi-ideal, then $\mu(0)$ is a minimal element of $\mu(X)$,*
(ii) *if μ is a fuzzy left semi-ideal, then $\mu(0)$ is a maximal element of $\mu(X)$.*

Proof. (i) Assume μ is a fuzzy right semi-ideal of a BCK-algebra $(X, *, 0)$. Then

$$\mu\big((x * z) * (y * z)\big) \leq \mu(x * y) \tag{3.1}$$

for all $x, y, z \in X$. If we let $z := x$ in (3.1), then

$$\begin{aligned} \mu(0) &= \mu(0 * (y * x)) \\ &= \mu\big((x * x) * (y * x)\big) \\ &\leq \mu(x * y). \end{aligned} \tag{3.2}$$

If we let $y := 0$ in (3.2), then by Theorem 2.1(v), we obtain

$$\mu(0) \leq \mu(x * 0) = \mu(x).$$

Hence $\mu(0)$ is a minimal element of $\mu(X)$.

(ii) Assume μ is a fuzzy left semi-ideal of a BCK-algebra $(X, *, 0)$. Then

$$\mu\big((z * x) * (z * y)\big) \leq \mu(x * y) \tag{3.3}$$

for all $x, y, z \in X$. If we let $x := 0$ in (3.3), then

$$\begin{aligned} \mu(0) &= \mu(0 * y) \\ &\geq \mu\big((z * 0) * (z * y\big) \\ &= \mu\big(z * (z * y)\big). \end{aligned} \tag{3.4}$$

If we let $y := z$ in (3.4), then by Theorem 2.1(v), we obtain

$$\mu(0) \geq \mu\big(z * (z * z)\big) = \mu(z * 0) = \mu(z).$$

Hence $\mu(0)$ is a maximal element of $\mu(X)$. □

Theorem 3.7. *Let $(X, *, 0)$ be a BCK-algebra and $\mu : (X, *) \to [0, 1]$ be a fuzzy set. The following hold:*

(i) *if μ is a fuzzy reverse right semi-ideal, then $\mu(0)$ is a maximal element of $\mu(X)$,*
(ii) *if μ is a fuzzy reverse left semi-ideal, then $\mu(0)$ is a minimal element of $\mu(X)$.*

Proof. (i) Assume μ is a fuzzy reverse right semi-ideal of a BCK-algebra $(X, *, 0)$. Then

$$\mu((x * z) * (y * z)) \geq \mu(x * y) \tag{3.5}$$

for all $x, y, z \in X$. If we let $z := x$ in (3.5), then

$$\begin{aligned} \mu(0) &= \mu(0 * (y * x)) \\ &= \mu((x * x) * (y * x)) \\ &\geq \mu(x * y). \end{aligned} \tag{3.6}$$

If we let $y := 0$ in (3.6), then by Theorem 2.1(v), we obtain

$$\mu(0) \geq \mu(x * 0) = \mu(x).$$

Hence $\mu(0)$ is a maximal element of $\mu(X)$.

(ii) Assume μ is a fuzzy left semi-ideal of a BCK-algebra $(X, *, 0)$. Then

$$\mu((z * x) * (z * y)) \geq \mu(x * y) \tag{3.7}$$

for all $x, y, z \in X$. If we let $x := 0$ in (3.7), then

$$\begin{aligned} \mu(0) &= \mu(0 * y) \\ &\leq \mu((z * 0) * (z * y)) \\ &= \mu(z * (z * y)). \end{aligned} \tag{3.8}$$

If we let $y := z$ in (3.8), then by Theorem 2.1(v), we obtain

$$\mu(0) \leq \mu(z * (z * z)) = \mu(z * 0) = \mu(z).$$

Hence $\mu(0)$ is a minimal element of $\mu(X)$. □

Corollary 3.8. *Let $(X, *, 0)$ be a BCK-algebra. If $\mu : (X, *) \to [0, 1]$ is a fuzzy semi-ideal or a fuzzy reverse semi-ideal, then μ is a constant fuzzy membership function.*

Proof. We conclude with Theorems 3.6 and 3.7. □

Theorem 3.9. *Let X be a groupoid. Then a fuzzy subset μ in X is a fuzzy subgroupoid of X if and only if, for each $t \in [0, 1]$, $\mu_t = \{x \in X \mid \mu(x) \geq t\}$ is a subgroupoid of X when $\mu_t \neq \emptyset$.*

Proof. Suppose μ is a fuzzy subgroupoid of X. For any $x, y \in \mu_t$ where $t \in [0, 1]$ and $\mu_t \neq \emptyset$, we have

$$\mu(x * y) \geq \min\{\mu(x), \mu(y)\} \geq t,$$

which implies $x * y \in \mu_t$, and so μ_t is a subgroupoid of X.

Conversely, for any x, y in X, denote $t = \min\{\mu(x), \mu(y)\}$. Then by the hypothesis μ_t is a subgroupoid of X, which implies that $x * y \in \mu_t$, so that $\mu(x * y) \geq t = \min\{\mu(x), \mu(y)\}$. This says that μ is a fuzzy subgroupoid of X. The proof is complete. □

Theorem 3.10. *Any subgroupoid of X can be realized as a level subgroupoid of some fuzzy subgroupoid of X.*

Proof. Let A be a subgroupoid of a X and let μ be a fuzzy subset in X defined by

$$\mu(x) = \begin{cases} t, & \text{if } x \in A, \\ 0, & \text{if } x \notin A, \end{cases}$$

where t is a fixed number in $(0, 1)$, we shall prove that μ is a fuzzy subgroupoid of X. Let $x, y \in X$. If $x, y \in A$, then $x * y \in A$. It follows that $\mu(x) = \mu(y) = \mu(x * y) = t$, and that $\mu(x * y) = \min\{\mu(x), \mu(y)\}$. If $x, y \notin A$, then $\mu(x) = 0 = \mu(y)$. Thus $\mu(x * y) \geq 0 = \min\{\mu(x), \mu(y)\}$. Suppose $x \in A$ and $y \notin A$ (resp. $x \notin A$ and $y \in A$). Then $\mu(x) = t$ and $\mu(y) = 0$ (resp. $\mu(x) = 0$ and $\mu(y) = t$). Thus $\mu(x * y) \geq 0 = \min\{\mu(x), \mu(y)\}$. Hence in all cases μ is a fuzzy subgroupoid of X. For the fuzzy subgroupoid μ, obviously $\mu_t = A$. □

Proposition 3.11. *If a fuzzy subset μ in a BCK-algebra X is a fuzzy reverse right semi-ideal of X, then for each $t \in [0, 1]$, μ_t is a right semi-ideal of X, when $\mu_t \neq \emptyset$.*

Proof. Assume μ is a fuzzy reverse right semi-ideal of a BCK-algebra $(X, *, 0)$. Then

$$\mu\big((x * z) * (y * z)\big) \geq \mu(x * y) \tag{3.9}$$

for all $x, y, z \in X$.

If $x * y \in \mu_t$, $z \in X$, then $\mu(x * y) \geq t$. Thus $\mu\big((x * z) * (y * z)\big) \geq t$. Then $(x * z) * (y * z) \in \mu_t$. □

Proposition 3.12. *If a fuzzy subset μ in a BCK-algebra X is a fuzzy reverse left semi-ideal of X, then for each $t \in [0,1]$, μ_t is a left semi-ideal of X, when $\mu_t \neq \emptyset$.*

Proof. Assume μ is a fuzzy reverse left semi-ideal of a BCK-algebra $(X, *, 0)$. Then

$$\mu\big((z*x)*(z*y)\big) \geq \mu(x*y) \tag{3.10}$$

for all $x, y, z \in X$.

If $x*y \in \mu_t$, $z \in X$, then $\mu(x*y) \geq t$. Thus $\mu\big((z*x)*(z*y)\big) \geq t$. Then $(z*x)*(z*y) \in \mu_t$. □

Given a groupoid $(X, *)$ and a poset $(X, \leq)$, and let $\mu : (X, *) \to [0,1]$ be a fuzzy subset of $(X, *)$. We define a set $P_\mu(*; \leq)$ as follows:

$$P_\mu(*; \leq) := \{y * x \,|\, \mu(x) \leq \mu(y)\}.$$

Note that $(X, *) \neq P_\mu(*; \leq)$, i.e., $P_\mu(*; \leq)$ is only a set, not containing an algebraic structure. We have $\mu(x) \leq \mu(x)$ for each $x \in X$ whence $P_\mu(*; \leq)$ always contains the non-empty subset $\{x * x \,|\, x \in X\}$.

A groupoid $(X, *)$ is said to be a pogroupoid [17] if (i) $x*y \in \{x, y\}$; (ii) $x*(y*x) = y*x$; (iii) $(x*y)*(y*z) = (x*y)*z$ for all $x, y, z \in X$.

Let $\mu : X \to [0,1]$ be a fuzzy subset. Define a binary operation $\odot$ on X by

$$x \odot y = \begin{cases} x & \text{if } \mu(x) > \mu(y), \\ y & \text{if } \mu(x) \leq \mu(y). \end{cases}$$

for all $x, y \in X$. We call such a groupoid $(X, \odot)$ a selective groupoid induced by a fuzzy subset μ, and we denote it by $(X, \odot)_\mu$.

Theorem 3.13. ([10]) *Every selective groupoid induced by a fuzzy subset is a pogroupoid.*

As we have seen [19], if we define a relation $\leq_\mu$ on X by

$$x \leq_\mu y \iff y \odot x = y = x \odot y \tag{3.11}$$

where $(X, \odot)_\mu$ is a pogroupoid induced by μ. Then $(X, \leq_\mu)$ is a poset, called an induced poset by a fuzzy subset μ.

Note that, using the notion of the pogroupoid, we can construct a partially ordered set on a set X by a fuzzy subset $\mu : X \to [0,1]$.

Example 3.14. *Let $X := \{a, b, c, d\}$ be a set. Define a map $\mu : X \to [0,1]$ by $\mu(a) = \mu(b) < \mu(c) < \mu(d)$. Then we can construct a selective groupoid $(X, \odot)_\mu$ induced by μ in Table 1.*

TABLE 1. Selective groupoid $(X, \odot)_\mu$ induced by μ.

$\odot$	a	b	c	d
a	a	b	c	d
b	a	b	c	d
c	c	c	c	d
d	d	d	d	d

Using (3.11), we obtain the poset $(X, \leq_\mu)$ induced by a fuzzy subset μ in Figure 1.

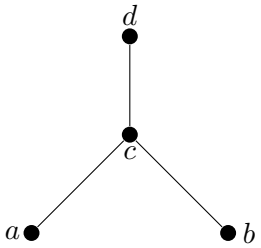

FIGURE 1. The poset $(X, \leq_\mu)$ induced by a fuzzy subset μ.

$$\begin{aligned} P_\mu(\odot; \leq_\mu) &= \{y \odot x \,|\, \mu(x) \leq \mu(y)\} \\ &= \{y \,|\, \mu(x) \leq \mu(y)\} \\ &= \{a, b, c, d\} = X. \end{aligned}$$

If we define another fuzzy subset $\nu : X \to [0, 1]$ by $\nu(a) < \nu(b) = \nu(c) < \nu(d)$, then we obtain the selective groupoid $(X, \odot)_\nu$ induced by ν in Table 2.

TABLE 2. Selective groupoid $(X, \odot)_\nu$ induced by ν.

$\odot$	a	b	c	d
a	a	b	c	d
b	b	b	c	d
c	c	b	c	d
d	d	d	d	d

Using (3.11), we obtain the poset $(X, \leq_\nu)$ induced by a fuzzy subset ν in Figure 2.

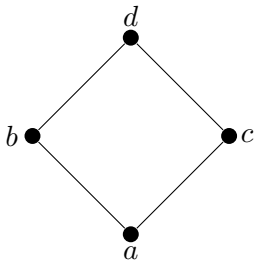

FIGURE 2. The poset $(X, \leq_\nu)$ induced by a fuzzy subset ν.

Indeed, $b \odot a = a \odot b = b$ implies $a \leq_\nu b, c \odot a = a \odot c = c$ implies $a \leq_\nu c$. Moreover, $d \odot \alpha = \alpha \odot d = d$ for all $\alpha \in X$ shows that $\alpha \leq_\nu d$.

$$\begin{aligned} P_\nu(\odot; \leq_\nu) &= \{y \odot x \,|\, \nu(x) \leq \nu(y)\} \\ &= \{y \,|\, \nu(x) \leq \nu(y)\} \\ &= \{a, b, c, d\} = X. \end{aligned}$$

Example 3.15. *Let $(X, \leq)$ be a poset with the following diagram:* Then its associated pogroupoid $(X, \cdot)$ in Table 3.

Define a map $\mu : X \to [0, 1]$ by $\mu(a) = \mu(b) < \mu(c) < \mu(d)$ as in Example 3.14. Then $\mu(d \cdot c) = \mu(c) < \mu(d) = \mu(c \cdot d)$, i.e., $d \rhd_\mu c$, and $\mu(d \cdot a) = \mu(a) < \mu(d) = \mu(a \cdot d)$. i.e., $d \rhd_\mu a$. Moreover, $\mu(a \cdot c) = \mu(c \cdot a)$, $\mu(a \cdot b) = \mu(b \cdot a)$, $\mu(b \cdot c) = \mu(c \cdot b)$ and $\mu(d \cdot b) = \mu(b \cdot d)$. Using the fact we obtain a poset $(X, \rhd_\mu)$ induced by μ in Figure 4.

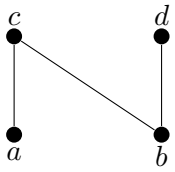

FIGURE 3. The poset $(X, \leq)$.

TABLE 3. The pogroupoid $(X, \cdot)$.

$\cdot$	a	b	c	d
a	a	b	c	d
b	a	b	c	d
c	c	c	c	d
d	a	d	c	d

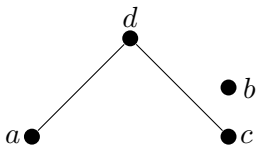

FIGURE 4. Poset $(X, \rhd_\mu)$ induced by μ.

$$\begin{aligned} P_\mu(\cdot; \rhd_\mu) &= \{y \cdot x \,|\, \mu(x) \leq \mu(y)\} \\ &= \{y \,|\, \mu(x) \leq \mu(y)\} \\ &= \{a, b, c, d\} = X. \end{aligned}$$

Proposition 3.16. *Let $(X, \odot)_\mu$ be a selective groupoid induced by μ. Define $x \leq_\mu y \iff y \odot x = y = x \odot y$. Then $P_\mu(\odot; \leq_\mu) = X$.*

Proof. Assume $\mu : X \to [0, 1]$ is a fuzzy subset of a selective groupoid X. Then

$$\begin{aligned} P_\mu(\odot; \leq_\mu) &= \{y \odot x \,|\, \mu(x) \leq \mu(y)\} \\ &= \{y \,|\, \mu(x) \leq \mu(y)\} \\ &= \{x \odot y \,|\, \mu(x) \leq \mu(y)\} \\ &= X. \end{aligned}$$

□

Theorem 3.17. *Let $(X, *, 0)$ be a BCK-algebra and let $\mu : (X, *) \to [0, 1]$ be a fuzzy subset. The following hold:*

(i) *if μ is a fuzzy right semi-ideal, then $P_\mu(*; \leq) = X$,*
(ii) *if μ is a fuzzy left semi-ideal, then $0 \in P_\mu(*; \leq)$,*
(iii) *if μ is a fuzzy reverse left semi-ideal, then $P_\mu(*; \leq) = X$,*
(iv) *if μ is a fuzzy reverse right semi-ideal, then $0 \in P_\mu(*; \leq)$.*

Proof. (i) Assume $\mu : (X, *) \to [0, 1]$ is a fuzzy right semi-ideal of a BCK-algebra $(X, *, 0)$. By Theorem 3.6(i), $\mu(0) \leq \mu(x)$ for all $x \in X$, then $x * 0 \in P_\mu(*; \leq)$. Thus $x = x * 0 \in P_\mu(*; \leq)$, i.e., $P_\mu(*; \leq) = X$.

(ii) Assume $\mu : (X, *) \to [0, 1]$ is a fuzzy left semi-ideal of a BCK-algebra $(X, *, 0)$. By Theorem 3.6(ii), $\mu(x) \leq \mu(0)$ for all $x \in X$, then $0 * x \in P_\mu(*, \leq)$. Thus $0 = 0 * x \in P_\mu(*; \leq)$.

(iii) Assume $\mu : (X, *) \to [0, 1]$ is a fuzzy reverse left semi-ideal of a BCK-algebra $(X, *, 0)$. By Theorem 3.7(i), $\mu(0) \leq \mu(x)$ for all $x \in X$, then $x * 0 \in P_\mu(*; \leq)$. Thus $x = x * 0 \in P_\mu(*; \leq)$, i.e., $P_\mu(*; \leq) = X$.

(iv) Assume $\mu : (X, *) \to [0, 1]$ is a fuzzy reverse right semi-ideal of a BCK-algebra $(X, *, 0)$. By Theorem 3.7(ii), $\mu(x) \leq \mu(0)$ for all $x \in X$, then $0 * x \in P_\mu(*, \leq)$. Thus $0 = 0 * x \in P_\mu(*; \leq)$. □

Theorem 3.18. *$\mu : (X, *) \to [0, 1]$ is a fuzzy right (left, resp.) compatible over a poset $(X, \leq)$, if and only if $P_\mu(*; \leq)$ is a right (left, resp.) semi-ideal of $(X, *)$.*

Proof. ($\Rightarrow$) If $x * y \in P_\mu(*; \leq)$, then $\mu(y) \leq \mu(x)$. Since μ is a fuzzy right (left, resp.) compatible over a poset $(X, \leq)$, we obtain $\mu(y * z) \leq \mu(x * z)$ for all $z \in X$. It follows that $(x * z) * (y * z) \in P_\mu(*; \leq)$.
($\Leftarrow$) If $\mu(x) \leq \mu(y)$, then $y * x \in P_\mu(*; \leq)$. Since $P_\mu(*; \leq)$ is a right (left, resp.) semi-ideal of $(X, *)$, we obtain $(y * z) * (x * z) \in P_\mu(*; \leq)$ $\big((z * y) * (z * x) \in P_\mu(*; \leq)\big)$ for all $z \in X$. Thus, $\mu(x * z) \leq \mu(y * z)$ $\big(\mu(z * x) \leq \mu(z * y)\big)$. □

Proposition 3.19. *Let $\mu : (X, *) \to [0, 1]$ be a full fuzzy partially ordered subgroupoid. If $P_\mu(*; \leq)$ is a subgroupoid of groupoid $(X, *)$, then $P_\mu(*; \leq)$ is a singleton set.*

Proof. Assume $a, b \in P_\mu(*; \leq)$. Since $P_\mu(*; \leq)$ is closed set, we get $a * b, b * a \in P_\mu(*; \leq)$. Hence $\mu(b) \leq \mu(a)$ and $\mu(a) \leq \mu(b)$. Since μ is full, we have $b \leq a$ and $a \leq b$, and so $a = b$. □

Theorem 3.20. *Let $\mu : (X, *) \to [0, 1]$ be a full fuzzy partially ordered subgroupoid and $(X, *)$ be medial and compatible over a poset $(X, \leq)$. Then $P_\mu(*; \leq)$ is a singleton set.*

Proof. Assume $a, b \in P_\mu(*; \leq)$. Then there are $x, y, r, s \in X$ such that $a = x * y, \mu(y) \leq \mu(x)$ and $b = r * s$, $\mu(s) \leq \mu(r)$. Since μ is full, we obtain $y \leq x$ and $s \leq r$. Since $(X, *)$ is compatible over a poset $(X, \leq)$ and μ is an order preserving map, we obtain $y * s \leq x * r$ and $\mu(y * s) \leq \mu(x * r)$. Hence $a * b = (x * y) * (r * s) = (x * r) * (y * s) \in P_\mu(*; \leq)$. Thus, $\mu(b) \leq \mu(a)$. Since μ is full, we obtain $b \leq a$. By a similar argument we get $b * a \in P_\mu(*; \leq)$, and so $a \leq b$. Therefore, $a = b$ and hence $P_\mu(*; \leq)$ is a singleton set. □

A groupoid $(X, *)$ is said to be fuzzy fully dominated by a poset $(X, \leq)$ induced by fuzzy subset μ if $P_\mu(*; \leq) = X$, i.e., for any $z \in X$, there exist $x, y \in X$ such that $z = y * x$ and $\mu(x) \leq \mu(y)$.

Theorem 3.21. *(i) Every left zero semigroup (i.e., $y * x = y$, for all $x, y \in X$) is fuzzy fully dominated.*
(ii) Every idempotent groupoid (i.e., $x * x = x$, for all $x \in X$) is fuzzy fully dominated.

Proof. (i) Let $(X, *)$ be a left zero semigroup. Since $(X, *)$ is a left zero semigroup, we have $y * x = y$ for all $x, y \in X$. Since $\mu(x) \leq \mu(x)$ for all $x \in X$, we obtain $P_\mu(*; \leq) = \{y * x \,|\, \mu(x) \leq \mu(y)\} = \{y \,|\, \mu(x) \leq \mu(y)\} = X$.
(ii) Assume that $x \in X$. Since $\mu(x) \leq \mu(x)$ for all $x \in X$, we obtain $x = x * x \in P_\mu(*; \leq)$. Thus $P_\mu(*; \leq) = X$. □

4. Conclusions

In this study, we explored the interplay between groupoids $(X, *)$, posets $(X, \leq)$ and fuzzy subset μ through the lens of the set $P_\mu(*; \leq) = \{y*x \mid \mu(x) \leq \mu(y)\}$. For a left zero semigroup, or an idempotent groupoid, the set $P_\mu(*; \leq)$ coincides with X, revealing a deep algebraic connection where the groupoid operation is fuzzy fully determined by the poset structure and them are fuzzy fully dominated by their underlying posets, meaning the groupoid operation is entirely governed by the order relation. As a direction of research, extend these results to BE-algebras, pseudo-BCK algebras, or quantum B-algebras, where implications and order relations play a central role, and investigate whether $P_\mu(*; \leq)$ induces a topological closure operator or a Galois connection between the lattice of groupoids and the lattice of posets on X. This work opens doors to unifying order theory and algebraic structures, with potential applications in logic, combinatorics, and theoretical computer science. The interplay between groupoids, posets and fuzzy subset μ via $P_\mu(*; \leq)$ offers a rich framework for future research, especially in non-classical algebras and ordered algebraic structures.

Acknowledgments The authors would like to thank referees for some very helpful comments in improving several aspects of this paper.

References

[1] S. S. Ahn and Y. H. Kim, *Some constructions of implicative/commutative d-algebras*, Bull. Korean Math. Soc., **46(1)** (2009), 147–153.

[2] J. Ahsan, J. N. Mordeson and M. Shabir, *Fuzzy Semirings with Applications to Automata Theory*, Springer, New York, 1971.

[3] R. H. Bruck, *A Survey of Binary Systems*, Springer, New York, 1971.

[4] O. Borůvka, *Foundations of the Theory of Groupoids and Groups*, John Wiley & Sons, New York, 1976.

[5] J. S. Han, H. S. Kim and J. Neggers, *On linear fuzzifications of groupoids with special emphasis on BCK-algebras*, Journal of Intellingent and Fuzzy Systems, **24(1)** (2013), 105–110.

[6] Y. Huang, *BCI-algebras*, Science Press, Beijing, 2006.

[7] I. H. Hwang, H. S. Kim and J. Neggers, *Some implicativities for groupoids and BCK-algebras*, Mathematics **7(10)** (2019), 973; https://doi.org/10.3390/math7100973

[8] A. Iorgulescu, *Algebras of Logic as BCK-algebras*, Editura ASE, Bucharest, 2008.

[9] K. Iséki and S. Tanaka, *An introduction to theory of BCK-algebras*, Sci. Math. Jpn., **23** (1975), 1–26.

[10] Y. H. Kim, H. S. Kim and J. Neggers, *Selective groupoids and frameworks induced by fuzzy subsets*, J. Fuzzy Syst., Iran, **14(3)** (2017), 151–160.

[11] H. S. Kim and J. Neggers, *Fuzzy pogroupoids*, Information Sciences, **175(1-2)** (2005), 108–119.

[12] H. S. Kim, J. Neggers and S. S. Ahn, *On pre-commutative algebras*, Mathematics **7(4)** (2019), 336; https://doi.org/10.3390/math7040336

[13] Y. L. Liu, H. S. Kim and J. Neggers, *Hyperfuzzy subsets and subgroupoids*, Journal of Intelligent and Fuzzy Systems, **33(3)** (2017), 1553–1562.

[14] Y. L. Liu, H. S. Kim and J. Neggers, *Some special elements and pseudo inverse functions in groupoids*, Mathematics **7(2)** (2019), 173; https://doi.org/10.3390/math7020173

[15] J. Meng and Y. B. Jun, *BCK-algebras*, Kyungmoon Sa, Seoul, 1994.

[16] J. N. Mordeson and D. S. Malik, *Fuzzy Commutative Algebras*, World Scientific, Singapore, 1998.

[17] J. Neggers, *Partially ordered sets and groupoids*, Kyungpook Math. J., **16** (1976), 7–20.

[18] J. Neggers and H. S. Kim, *On d-algebras*, Math. Slovaca, **49(1)** (1999), 19–25.

[19] J. Neggers and H. S. Kim, *Modular posets and semigroups*, Semigr. Forum, **53** (1996), 57–62.

[20] A. Rezaei and H. S. Kim, *On groupoid-poset structures*, New Mathematics and Natural Computation, In Press, DOI: 10.1142/S1793005728500664

[21] A. Rezaei, C. Park and H. S. Kim, *Some orders in groupoids and its applications to fuzzy groupoids*, Transactions on Fuzzy Sets and Systems, **4(1)** (2023), 4–13.

[22] A. Rosenfeld, *Fuzzy groups*, Journal of Mathematical Analysis and Applications, **35(3)** (1971), 512–517.

[23] L. A. Zadeh, *Fuzzy sets*, Information and Control, **8(3)** (1965), 338–353.