DOI: 10.22098/jhs.2026.18976.1155

Journal of Hyperstructures, Article in Press

Journal Homepage: https://jhs.uma.ac.ir/

Research Paper

ANALYSIS OF 2-CONFORMAL VECTOR FIELDS ON WARPED PRODUCTS

GHODRATALLAH FASIHI-RAMANDI[1,*], MOSAYEB ZOHREHVAND[2] AND MAJID ALI CHOUDHARY[3]

[1]Department of Pure Mathematics, Imam Khomeini International University, Qazvin, Iran, fasihi@sci.ikiu.ac.ir

[2]Department of Mathematics, Malayer University, Malayer, Iran, m.zohrehvand@malayeru.ac.ir

[3]Department of Mathematics, Maulana Azad National Urdu University, Hyderabad, India, majid_alichoudhary@yahoo.co.in

ARTICLE INFO

Article history:
Received: 04 December 2025
Accepted: 27 September 2026
Communicated by Dariush Latifi

Keywords:
Warped product manifolds
Semi-Riemannian geometry
2-conformal vector fields
space-time

MSC:
53C21; 53C44

ABSTRACT

In this paper, we define 2-conformal vector fields (CVFs) within a Riemannian manifold (RM) and investigate their geometry in both standard static space-time (SSST) and warped product manifolds (WPM). We derive the conditions under which the 2-CVFs are parallel and establish curvature-related results for WPM in terms of 2-CVFs. We characterize 2-CVFs on static and Robertson-Walker space-times. Finally, the relation between such a vector field (VF) and hyperbolic Ricci solitons (HRS) is considered.

Abbreviations

This manuscript employs the following abbreviations:

CVFs	Conformal Vector Fields
WPM	Warped Product Manifolds
HRS	Hyperbolic Ricci Solitons
RS	Ricci Solitons
VF	Vector Fields
RM	Riemannian Manifolds
SSST	Standard Static space-times

∗Address correspondence to G. Fasihi-Ramandi; Department of Pur Mathematics, Faculty of Sciences, Imam Khomeini International University, Qazvin, Iran, E-mail: fasihi@sci.ikiu.ac.ir.

This work is licensed under a Creative Commons Attribution-NonCommercial 4.0 International License.
Copyright © 2026 The Author(s). Published by University of Mohaghegh Ardabili.

1. Introduction

Widely explored on Riemannian and semi-Riemannian manifolds, geometric vector fields have a recognized geometrical and physical interpretation. Conformal, Killing, and projective vector fields are among the most famous geometric vector fields, extensively debated in physics and geometry. In general relativity, these vector fields represent fundamental spacetime symmetries and conservation laws. For instance, a time-like Killing vector field corresponds to energy conservation, while conformal vector fields describe the symmetries of massless fields.

Killing vector fields have long been generalized mathematically, covering various scales and purposes [6, 7, 8, 13]. The notion of 2-Killing vector fields is defined, and some geometric results are obtained in [17], while the authors of [18] investigated it on WPM and characterized this structure on both generalized and standard static Robertson-Walker space-times. This research introduces the idea of a 2-CVF and considers its geometry on WPM, proving similar results as in [18].

A vector field $\mathbf{Z} \in \mathcal{X}(M)$ is referred to as a 2-CVF on a semi-Riemannian manifold (M, g) if

$$L_{\mathbf{Z}} L_{\mathbf{Z}} g = 2\sigma g,$$

for some smooth function σ (which will be named the conformal factor) on M, where L is the Lie derivative operator and $\mathcal{X}(M)$ represents the Lie algebra of smooth vector fields on M. If σ is identically zero, $\mathbf{Z}$ is a 2-Killing vector field. Also, when σ is a constant, the 2-CVF $\mathbf{Z}$ is called homothetic; otherwise, it is said to be a non-homothetic field. Obviously, every CVF is a 2-CVF. Hence, a proper 2-conformal field is defined as a 2-conformal field which is not a CVF.

Example 1.1. Consider the 2-dimensional Euclidean space M with the standard coordinate system (x, y) and its canonical metric g_{can}. It can be readily verified that $\mathbf{Z} = \alpha\partial_x + \beta\partial_y$ is a homothetic 2-CVF, meaning

$$L_{\mathbf{Z}} L_{\mathbf{Z}} g = 2cg,$$

if and only if

$$\alpha\alpha_{xx} + 2\alpha_x^2 = \beta\beta_{yy} + 2\beta_y^2 = c.$$

The subsequent proposition is presented.

Proposition 1.2. *Assume $\mathbf{Z} \in \mathcal{X}(M)$ is a 2-CVF on (M, g). Then*

$$\begin{aligned} &g(\nabla_{\mathbf{Z}}\nabla_X\mathbf{Z} - \nabla_{[\mathbf{Z},X]}\mathbf{Z}, Y) + g(X, \nabla_{\mathbf{Z}}\nabla_Y\mathbf{Z} - \nabla_{[\mathbf{Z},Y]}\mathbf{Z}) \\ &+ 2g(\nabla_X\mathbf{Z}, \nabla_Y\mathbf{Z}) = 2\sigma g(X, Y), \quad \forall X, Y \in \mathcal{X}(M). \end{aligned} \tag{1.1}$$

Proof. In [17], it has been demonstrated that for a $\mathbf{Z} \in \mathcal{X}(M)$ on (M, g), we have

$$\begin{aligned} (L_{\mathbf{Z}} L_{\mathbf{Z}} g)(X, Y) =& g(\nabla_{\mathbf{Z}}\nabla_X\mathbf{Z} - \nabla_{[\mathbf{Z},X]}\mathbf{Z}, Y) \\ &+ g(X, \nabla_{\mathbf{Z}}\nabla_Y\mathbf{Z} - \nabla_{[\mathbf{Z},Y]}\mathbf{Z}) + 2g(\nabla_X\mathbf{Z}, \nabla_Y\mathbf{Z}). \end{aligned}$$

Hence, if $\mathbf{Z}$ is a 2-CVF on (M, g), it must satisfy (1.1). □

This direct result aids our further discussion.

Corollary 1.3. $\mathbf{Z} \in \mathcal{X}(M)$ *is* 2*-conformal iff*

$$Rm(\mathbf{Z}, X, \mathbf{Z}, X) = g(\nabla_X \mathbf{Z}, \nabla_X \mathbf{Z}) + g(\nabla_X \nabla_{\mathbf{Z}} \mathbf{Z}, X) - \sigma g(X, X), \tag{1.2}$$

$\forall X \in \mathcal{X}(M)$. *Here,* Rm *denotes the* $(0,4)$*-type Riemannian curvature tensor of* g.

Proof. Putting $Y = X$ in (1.1), we obtain

$$g(\nabla_{\mathbf{Z}} \nabla_X \mathbf{Z} - \nabla_{[\mathbf{Z},X]} \mathbf{Z}, X) + g(\nabla_X \mathbf{Z}, \nabla_X \mathbf{Z}) = \sigma g(X, X).$$

Now, we can write

$$\begin{aligned} Rm(\mathbf{Z}, X, \mathbf{Z}, X) &= -g(R(\mathbf{Z}, X)(\mathbf{Z}), X) = g(-\nabla_{\mathbf{Z}} \nabla_X \mathbf{Z} + \nabla_X \nabla_{\mathbf{Z}} \mathbf{Z} + \nabla_{[\mathbf{Z},X]} \mathbf{Z}, X) \\ &= -g(\nabla_{\mathbf{Z}} \nabla_X \mathbf{Z} - \nabla_{[\mathbf{Z},X]} \mathbf{Z}, X) + g(\nabla_X \nabla_{\mathbf{Z}} \mathbf{Z}, X) \\ &= g(\nabla_X \mathbf{Z}, \nabla_X \mathbf{Z}) + g(\nabla_X \nabla_{\mathbf{Z}} \mathbf{Z}, X) - \sigma g(X, X). \end{aligned}$$

Putting $X := X + Y$ in (1.2), the converse is obtained. □

The symmetry of the above formula shows that $\mathbf{Z}$ is a 2-CVF iff

$$g(\nabla_X \mathbf{Z}, \nabla_X \mathbf{Z}) + g(\nabla_{\mathbf{Z}} \nabla_X \mathbf{Z} - \nabla_{[\mathbf{Z},X]} \mathbf{Z}, X) = \sigma g(X, X), \tag{1.3}$$

for every $X \in \mathcal{X}(M)$.

2. Warped product manifolds

This section contains fundamental definitions and mathematics pertaining to WPM. For more details, one can consult [16].

Consider (M_1^m, g_1, ∇^1) and (M_2^n, g_2, ∇^2) as two semi-Riemannian manifolds endowed with metric tensors g_i and corresponding ∇^i, where $i = 1, 2$. The functions $\pi_1 : M_1 \times M_2 \to M_1$ and $\pi_2 : M_1 \times M_2 \to M_2$ serve as the natural projection maps from the product manifold $M_1 \times M_2$ onto M_1 and M_2, respectively. If $f : M_1 \to (0, \infty)$ is a positive, smooth function that takes real values on M_1, then the WPM $M_1 \times_f M_2$ is constituted by endowing $M_1 \times M_2$ with g specified as

$$g = \pi_1^*(g_1) \oplus (f \circ \pi_1)^2 \pi_2^*(g_2),$$

In this context, $*$ represents the pull-back operator for tensors, and f is identified as the warping function of the WPM $M_1 \times_f M_2$. It should be noted that when $f = 1$, $M_1 \times_f M_2$ reduces to the standard Cartesian product manifold $M_1 \times M_2$. Moreover, it is clear that for every $q \in M_2$, $M_1 \times \{q\}$ remains isometric to M_1. Likewise, for any $p \in M_1$, the set $\{p\} \times M_2$ is homothetic to M_2.

Here is a well-known proposition.

Proposition 2.1. [5] *Consider* $M_1 \times_f M_2$ *a semi-Riemannian WPM with* $f > 0$. *Then*

1. $\nabla_{X_1} Y_1 = \nabla^1_{X_1} Y_1 \in \mathcal{X}(M_1)$,
2. $\nabla_{X_1} Y_2 = \nabla_{Y_2} X_1 = \dfrac{X_1(f)}{f} Y_2$,
3. $\nabla_{X_2} Y_2 = -f g_2(X_2, Y_2) \nabla^1 f + \nabla^2_{X_2} Y_2$,

$\forall X_i, Y_i \in \mathcal{X}(M_i)$, *with* $i = 1, 2$, *where* $\nabla^1 f$ *stands for the gradient of* f.

Using Proposition 1.2, and the following theorem, one can obtain the essential and adequate circumstances for $(\mathbf{Z}_1, \mathbf{Z}_2) \in \mathcal{X}(M_1 \times_f M_2)$ to be a 2-CVF on the Riemannian WPM $M_1 \times_f M_2$. The subsequent theorem states.

Theorem 2.2. [18] *Let $\mathbf{Z} = (\mathbf{Z}_1, \mathbf{Z}_2) \in \mathcal{X}(M_1 \times_f M_2)$ on $M_1 \times_f M_2$. Then*

$$\begin{aligned} L_{\mathbf{Z}} L_{\mathbf{Z}} g = &(L^1_{\mathbf{Z}_1} L^1_{\mathbf{Z}_1} g_1)(X_1, Y_1) + f^2 (L^2_{\mathbf{Z}_2} L^2_{\mathbf{Z}_2} g_2)(X_2, Y_2) \\ &+ 4f \mathbf{Z}_1(f)(L^2_{\mathbf{Z}_2} g_2)(X_2, Y_2) + 2f \mathbf{Z}_1(\mathbf{Z}_1(f)) g_2(X_2, Y_2) \\ &+ 2\mathbf{Z}_1(f)\mathbf{Z}_1(f) g_2(X_2, Y_2), \end{aligned}$$

$\forall X = X_1 + X_2, Y = Y_1 + Y_2 \in \mathcal{X}(M_1 \times_f M_2)$. Here L^1 and L^2 are the Lie derivative operators on (M_1, g_1) and (M_2, g_2), respectively.

The corollaries below directly follow from the preceding theorem.

Corollary 2.3. *If $\mathbf{Z} = (\mathbf{Z}_1, \mathbf{Z}_2) \in \mathcal{X}(M_1 \times_f M_2)$ on $M_1 \times_f M_2$ is a 2-conformal field on $M_1 \times_f M_2$, then $\mathbf{Z}_1$ is a 2-CVF on M_1.*

Corollary 2.4. *Assume $\mathbf{Z} \in \mathcal{X}(M_1 \times_f M_2)$ and suppose that $\mathbf{Z}_1$ and $\mathbf{Z}_2$ are 2-CVFs on M_1 and M_2, respectively. Then, $\mathbf{Z} = (\mathbf{Z}_1, \mathbf{Z}_2)$ is a 2-CVF on $M_1 \times_f M_2$ if $\mathbf{Z}_1(f) = 0$.*

Corollary 2.5. *Let $\mathbf{Z} \in \mathcal{X}(M_1 \times_f M_2)$ and assume that $\mathbf{Z}_1$ and $\mathbf{Z}_2$ are, respectively, 2-conformal homothetic vector fields on M_1 and M_2 (with the same factor $c > 0$). Then, $\mathbf{Z} = (\mathbf{Z}_1, \mathbf{Z}_2)$ is a 2-conformal homothetic VF on $M_1 \times_f M_2$ iff*
a) $\mathbf{Z}_1(f) = 0$, or
b) On M_2, $\mathbf{Z}_2$ is a homothetic VF with a homothetic factor (i.e., $L_{\mathbf{Z}_2} g_2 = \sqrt{c} g_2$) satisfying

$$f\mathbf{Z}_1(\mathbf{Z}_1(f)) + \mathbf{Z}_1(f)\mathbf{Z}_1(f) + f\sqrt{c}\mathbf{Z}_1(f) = 0.$$

Corollary 2.6. *$\mathbf{Z} = (\mathbf{Z}_1, \mathbf{Z}_2) \in \mathcal{X}(M_1 \times_f M_2)$ on $M_1 \times_f M_2$ is a 2-conformal field if any of the preceding criteria is true:*

(1) *$\mathbf{Z}_i$ is a 2-CVF on M_i, $i = 1, 2$, and $\mathbf{Z}_1(f) = 0$.*
(2) *$\mathbf{Z} = (0, \mathbf{Z}_2)$ and $\mathbf{Z}_2$ is a 2-CVF on M_2.*

The following theorem can also be proved.

Theorem 2.7. *Assume $\mathbf{Z} \in \mathcal{X}(M_1 \times_f M_2)$ on $M_1 \times_f M_2$ such that $\nabla_{\mathbf{Z}} \mathbf{Z} = 0$. Then*

(1) *$\mathbf{Z} = (\mathbf{Z}_1, \mathbf{Z}_2)$ is parallel, if $\mathbf{Z}_i$ is a 2-CVF with conformal factor σ, $\mathrm{Ric}^i(\mathbf{Z}_i, \mathbf{Z}_i) \leq -\dim(M_i).\sigma, \quad i = 1, 2$, and f is a positive constant.*
(2) *$\mathbf{Z} = \mathbf{Z}_1$ is parallel if $\mathbf{Z}_1$ is a 2-CVF, $\mathrm{Ric}^1(\mathbf{Z}_1, \mathbf{Z}_1) \leq -m\sigma$ and $\mathbf{Z}_1(f) = 0$.*

By Ric^i we mean the Ricci curvature tensor of manifold (M_i, g_i).

Proof. Let $\{e_j\}_{j=1}^m$ be an orthonormal frame for T_pM_1 and $\{e_{m+j}\}_{j=1}^n$ is an orthonormal frame in T_qM_2. Then

$$\{e_1, \cdots, e_m, \frac{1}{f}e_{m+1}, \cdots, \frac{1}{f}e_{m+n}\},$$

is an orthonormal frame of $T_{(p,q)}(M_1 \times_f M_2)$. Thus for any $\mathbf{Z} = (\mathbf{Z}_1, \mathbf{Z}_2) \in \mathcal{X}(M_1 \times_f M_2)$, we can write

$$Tr(g(\nabla \mathbf{Z}, \nabla \mathbf{Z})) = \sum_{j=1}^{m} g(\nabla_{e_j} \mathbf{Z}, \nabla_{e_j} \mathbf{Z}) + \frac{1}{f^2} \sum_{j=m+1}^{m+n} g(\nabla_{e_j} \mathbf{Z}, \nabla_{e_j} \mathbf{Z}). \tag{2.1}$$

Proposition 2.1 shows that the first term of (2.1) is provided by what follows

$$\begin{aligned}\sum_{j=1}^{m} g(\nabla_{e_j}\mathbf{Z},\nabla_{e_j}\mathbf{Z}) &= \sum_{j=1}^{m} g(\nabla^1_{e_j}\mathbf{Z}_1 + e_j(\ln f)\mathbf{Z}_2, \nabla^1_{e_j}\mathbf{Z}_1 + e_j(\ln f)\mathbf{Z}_2)\\ &= \sum_{j=1}^{m} g(\nabla^1_{e_j}\mathbf{Z}_1,\nabla^1_{e_j}\mathbf{Z}_1) + \sum_{j=1}^{m} g(e_j(\ln f)\mathbf{Z}_2, e_j(\ln f)\mathbf{Z}_2)\\ &= Tr(g_1(\nabla^1\mathbf{Z},\nabla^1\mathbf{Z})) + ||\mathbf{Z}_2||_2^2\sum_{j=1}^{m}(e_j(\ln f))^2\\ &= Tr(g_1(\nabla^1\mathbf{Z},\nabla^1\mathbf{Z})) + ||\mathbf{Z}_2||_2^2\frac{||\nabla f||_1^2}{f^2}, \qquad (2.2)\end{aligned}$$

and for the second term of (2.1) we have

$$\begin{aligned}\frac{1}{f^2}\sum_{j=m+1}^{m+n} g(\nabla_{e_j}\mathbf{Z},\nabla_{e_j}\mathbf{Z}) &= \frac{1}{f^2}\sum_{j=m+1}^{m+n} g\big(\mathbf{Z}_1(\ln f)e_j + \nabla^2_{e_j}\mathbf{Z}_2\\ &\quad - fg_2(e_j,\mathbf{Z}_2)\nabla f, \mathbf{Z}_1(\ln f)e_j + \nabla^2_{e_j}\mathbf{Z}_2) - fg_2(e_j,\mathbf{Z}_2)\nabla f\big)\\ &= n\frac{(\mathbf{Z}_1(f))^2}{f^4} + \frac{1}{f^2}\sum_{j=m+1}^{m+n} g_2(\nabla^2_{e_j}\mathbf{Z}_2,\nabla^2_{e_j}\mathbf{Z}_2)\\ &\quad + ||\nabla f||_1^2\sum_{j=m+1}^{m+n}(g_2(e_j,\mathbf{Z}_2))^2. \qquad (2.3)\end{aligned}$$

Hence,

$$\frac{1}{f^2}\sum_{j=m+1}^{m+n} g(\nabla_{e_j}\mathbf{Z},\nabla_{e_j}\mathbf{Z}) = \frac{1}{f^2}Tr(g_2(\nabla^2\mathbf{Z}_2,\nabla^2\mathbf{Z}_2)) + \frac{n}{f^4}(\mathbf{Z}_1(f))^2 + ||\nabla f||_1^2||\mathbf{Z}_2||_2^2. \qquad (2.4)$$

Using the above formulas, we derive

$$\begin{aligned}Tr(g(\nabla\mathbf{Z},\nabla\mathbf{Z})) &= Tr(g_1(\nabla^1\mathbf{Z},\nabla^1\mathbf{Z})) + \frac{1}{f^2}Tr(g_2(\nabla^2\mathbf{Z}_2,\nabla^2\mathbf{Z}_2))\\ &\quad + ||\mathbf{Z}_2||_2^2||\nabla f||_1^2\left(1+\frac{1}{f^2}\right) + \frac{n}{f^4}(\mathbf{Z}_1(f))^2. \qquad (2.5)\end{aligned}$$

Now, let us assume $\mathbf{Z}_i$ is a 2-CVF and $\mathrm{Ric}^i(\mathbf{Z}_i,\mathbf{Z}_i) \leq -\dim(M_i).\sigma$. By taking the trace of equation (1.3) over an orthonormal frame on (M_i, g_i), and using the assumption $\nabla_{\mathbf{Z}_i}\mathbf{Z}_i = 0$, we obtain

$$\sum_j g_i(\nabla_{e_j}\mathbf{Z}_i,\nabla_{e_j}\mathbf{Z}_i) - \mathrm{Ric}^i(\mathbf{Z}_i,\mathbf{Z}_i) = \dim(M_i)\sigma.$$

Given the condition $\mathrm{Ric}^i(\mathbf{Z}_i,\mathbf{Z}_i) \leq -\dim(M_i).\sigma$, this implies that $\sum_j g_i(\nabla_{e_j}\mathbf{Z}_i,\nabla_{e_j}\mathbf{Z}_i) \leq 0$. Since the metric g_i is positive semi-definite, we must have $\nabla\mathbf{Z}_i = 0$, meaning $\mathbf{Z}_i$ is parallel with respect to g_i. Therefore

$$Tr(g_1(\nabla^1\mathbf{Z},\nabla^1\mathbf{Z})) = Tr(g_2(\nabla^2\mathbf{Z}_2,\nabla^2\mathbf{Z}_2)) = 0.$$

Thus for a positive constant function f (which implies $\nabla f = 0$ and $\mathbf{Z}_1(f) = 0$) we get

$$Tr(g(\nabla\mathbf{Z},\nabla\mathbf{Z})) = 0.$$

Hence $\mathbf{Z}$ is a parallel field regarding g. The last two assertions can be easily proved by the formula (2.5). □

2-CVFs on WPM also yield the subsequent outcome on the sectional curvature K of the underlying manifold.

Theorem 2.8. *Suppose $\mathbf{Z} \in \mathcal{X}(M_1 \times_f M_2)$ is a non-zero 2-CVF with non-positive conformal factor σ and $\nabla_{\mathbf{Z}}\mathbf{Z} = 0$. Then, for every $X \in \mathcal{X}(M_1 \times_f M_2)$, for the sectional curvature K, we have:*

$$K(\mathbf{Z}, X) \geq 0.$$

Proof. Since $\mathbf{Z}$ is a non-zero 2-CVF, then

$$\begin{aligned} Rm(\mathbf{Z}, X, \mathbf{Z}, X) &= g(\nabla_X \mathbf{Z}, \nabla_X \mathbf{Z}) + g(\nabla_X \nabla_{\mathbf{Z}} \mathbf{Z}, X) - \sigma g(X, X) \\ &= ||\nabla_X \mathbf{Z}||^2 + 0 - \sigma ||X||^2 \\ &= ||\nabla_X \mathbf{Z}||^2 - \sigma ||X||^2 \geq 0. \end{aligned}$$

Thus, for the sectional curvature $K(\mathbf{Z}, X)$, one can obtain the subsequent inequality.

$$K(\mathbf{Z}, X) = \frac{Rm(\mathbf{Z}, X, \mathbf{Z}, X)}{g(\mathbf{Z}, \mathbf{Z})g(X, X) - g(\mathbf{Z}, X)^2} = \frac{||\nabla_X \mathbf{Z}||^2 - \sigma ||X||^2}{||\mathbf{Z}||^2 ||X||^2 - g(\mathbf{Z}, X)^2} \geq 0,$$

where the denominator represents the square of the area of the parallelogram formed by X and $\mathbf{Z}$. □

Corollary 2.9. *If $\mathbf{Z} \in \mathcal{X}(M_1 \times_f M_2)$ is a non-trivial 2-CVF with non-positive conformal factor σ. If $\nabla_{\mathbf{Z}}\mathbf{Z}$ is parallel along a curve γ, then*

$$K(\mathbf{Z}, \gamma') \geq 0.$$

3. 2-CVFs of standard static space-time

Here, we apply our results in the context of a warped product space-time, recognized as the SSST. Consider an n-dimensional RM (M, g), and let $f : M \to (0, \infty)$ represent a smooth function. Therefore, $I \times M^{(n+1)}$ equipped with

$$\bar{g} = -f^2 dt^2 \oplus g$$

is described as a standard static space-time, denoted by $\bar{M} = I \times_f M$, with the Euclidean metric tensor on the interval $I = (c, d) \subseteq \mathbb{R}$ being represented by dt^2. In the context of physics, a standard static space-time is often considered as an extension of the Einstein universe [1, 2, 3, 4].

Theorem 3.1. *Suppose $\bar{M}$ is a SSST with $\bar{g}$. Assume $u : I \to \mathbb{R}$ is a smooth function. Then $\bar{\mathbf{Z}} = u\partial_t + \mathbf{Z}$, $\mathbf{Z} \in \mathcal{X}(M)$ is a 2-CVF on $\bar{M}$ with factor $\bar{\sigma} = (0, \sigma)$ if one of the subsequent criteria holds:*

1. *$\mathbf{Z}$ is conformal on M with conformal factor σ, $u = a$ and $f\mathbf{Z}(f) = b$, here $a, b \in \mathbb{R}$.*
2. *$\mathbf{Z}$ is conformal on M with conformal factor σ, $u = \sqrt[3]{at + b}$ and $\mathbf{Z}(f) = 0$, where $a, b \in \mathbb{R}$.*

Proof. To demonstrate this theorem, we employ Theorem 2.2. Given $\alpha, \beta \in I$ as smooth real-valued functions, the expressions $\bar{X} = \alpha\partial_t + X$ and $\bar{Y} = \beta\partial_t + Y$ represent smooth VFs on $\bar{M}$. Consequently, we obtain,

$$\begin{aligned} \big(\bar{L}_{\bar{\mathbf{Z}}} \bar{L}_{\bar{\mathbf{Z}}} \bar{g}\big)(\bar{X}, \bar{Y}) = (L_{\mathbf{Z}} L_{\mathbf{Z}} g)(X, Y) + f^2 (L^I_{u\partial_t} L^I_{u\partial_t} g_I)(\alpha\partial_t, \beta\partial_t) + 4f\mathbf{Z}(f)(L^I_{u\partial_t} g_I)(\alpha\partial_t, \beta\partial_t) \\ + 2f\mathbf{Z}(\mathbf{Z}(f)) g_I(\alpha\partial_t, \beta\partial_t) + 2\mathbf{Z}(f)\mathbf{Z}(f) g_I(\alpha\partial_t, \beta\partial_t). \end{aligned}$$

By straightforward computation we can easily deduce that

$$(L^I_{u\partial_t} g_I)(\alpha\partial_t, \beta\partial_t) = 2u' g_I(\alpha\partial_t, \beta\partial_t),$$
$$(L^I_{u\partial_t} L^I_{u\partial_t} g_I)(\alpha\partial_t, \beta\partial_t) = (2uu'' + 4u'^2) g_I(\alpha\partial_t, \beta\partial_t).$$

Hence

$$\begin{aligned}\big(\bar{L}_{\bar{\mathbf{Z}}}\bar{L}_{\bar{\mathbf{Z}}}\bar{g}\big)(\bar{X},\bar{Y}) = \\ (L_{\mathbf{Z}}L_{\mathbf{Z}}g)(X,Y) + f^2(2uu'' + 4u'^2) g_I(\alpha\partial_t, \beta\partial_t) + 8u' f\mathbf{Z}(f) g_I(\alpha\partial_t, \beta\partial_t) \\ + 2\mathbf{Z}(f\mathbf{Z}(f)) g_I(\alpha\partial_t, \beta\partial_t).\end{aligned} \tag{3.1}$$

$\mathbf{Z}$ is 2-conformal on M and u as described in (1) and (2) is a solution of the subsequent:

$$(2uu'' + 4u'^2) = 0.$$

Now, (3.1) can be rewritten as

$$\big(\bar{L}_{\bar{\mathbf{Z}}}\bar{L}_{\bar{\mathbf{Z}}}\bar{g}\big)(\bar{X},\bar{Y}) = (L_{\mathbf{Z}}L_{\mathbf{Z}}g)(X,Y) + 2\big(4u' f\mathbf{Z}(f) + \mathbf{Z}(f\mathbf{Z}(f))\big) g_I(\alpha\partial_t, \beta\partial_t) \tag{3.2}$$

Consequently, conditions (1) or (2) imply that $\bar{\mathbf{Z}}$ is a 2-CVF on $\bar{M}$. □

The next corollary follows right away.

Corollary 3.2. *Let $\bar{\mathbf{Z}} = u\partial_t + \mathbf{Z}$, $\mathbf{Z} \in \mathcal{X}(M)$, be a 2-CVF on $\bar{M}$. Then $\mathbf{Z}$ is a 2-CVF on M. Moreover, $u\partial_t$ is 2-conformal on I, if $\mathbf{Z}(f) = 0$.*

4. 2-CVFs on generalized Robertson-Walker

One of the most important space-time manifolds is the generalized Robertson-Walker space-time. Here, we characterize homothetic 2-CVFs in such space-times with positive homothetic factors.

Let $I = (a, b) \subset \mathbb{R}$ be a connected interval with Euclidean metric dt^2 and (M^n, g) is a RM, then $\bar{M} = I \times M$ with

$$\bar{g} = -dt^2 + f^2 g,$$

is referred to as a generalized Robertson-Walker space-time, where $f : (a, b) \to \mathbb{R}^+$ is a smooth function [14, 15].

As we mentioned before, for a VF $u\partial_t$ we have

$$(L^I_{u\partial_t} L^I_{u\partial_t} g_I)(\partial_t, \partial_t) = (2uu'' + 4u'^2) g_I(\partial_t, \partial_t).$$

As an easy application of Corollary 2.5, there is a subsequent outcome.

Proposition 4.1. *Suppose c is a positive constant. Then $\mathbf{Z} = u\partial_t + X$ is a homothetic 2-CVF on $\bar{M}$ with homothetic factor c if X is a homothetic CVF on (M, g) with factor $\sqrt{c}$ and the following equations hold*

$$uu'' + 2u'^2 = c,$$
$$ff'' + f'^2 = -\frac{2\sqrt{c} + u'}{u} ff'.$$

5. 2-conformal fields as potentials for hyperbolic Ricci solitons

A Ricci soliton (RS) is a semi-Riemannian manifold (M, g) equipped with a smooth vector field X that fulfills [11, 12]

$$L_X g + 2\mathrm{Ric} = 2\lambda g,$$

where $L_X g$ stands for the Lie derivative of g along X and λ is a real constant. It serves as a self-similar solution to the Ricci flow introduced by Hamilton. An RS is represented by (M, g, X, λ), where X is referred to as its potential field. The soliton is classified as:

- *steady* when $\lambda = 0$,
- *shrinking* when $\lambda > 0$, or
- *expanding* when $\lambda < 0$.

In addition, if there is an $f \in C^\infty(M)$ such that $X = \nabla f$, then (M, g, X, λ) is called a gradient RS.

Letting the soliton constant be a smooth function, in an attempt to generalize RSs, the notion of an almost RS is presented. Then, the notion of an h-almost RS is introduced. By an h-almost RS we mean (M, g) equipped with a smooth vector field X on M, satisfying

$$hL_X g + 2Ric = 2\lambda g,$$

where h and λ are smooth functions. Ricci and h-almost RSs have garnered significant interest. This is largely due to their elegant geometric properties and their diverse applications in various fields.

In contrast, in [10] the hyperbolic Ricci flow—a system of 2^{nd}-order nonlinear evolution PDEs—is explored. This flow preserves wave-like behaviour in the evolution of manifold metrics and curvature. Modelled after classical Ricci flow theory, it is governed by the subsequent evolution equation:

$$\frac{1}{2}\frac{\partial^2}{\partial t^2} g(t) = -2\mathrm{Ric}(g(t)), \quad g_0 = g(0), \quad \frac{\partial}{\partial t} g_{ij} = h_{ij}, \tag{5.1}$$

where h_{ij} is a symmetric 2-tensor field. Therefore, a self-similar solution of the hyperbolic Ricci flow is referred to as an HRS and is defined for the first time by the first author of the present paper as:

Definition 5.1. [9] (M^n, g) qualifies as an HRS if there is a vector field X on M with scalars μ and λ such that

$$L_X L_X g + \lambda L_X g + \mathrm{Ric} = \mu g, \tag{5.2}$$

where Ric is the Ricci curvature of M.

If X is a Killing VF, then the HRS reduces to an Einstein manifold. When X is a proper 2-Killing VF and $\lambda = \frac{1}{2}$, we reach an RS structure. Letting X be a 2-CVF, one can see that HRSs are related to h-almost RSs. If the potential field of a hyperbolic Ricci soliton is a 2-CVF, then the wave and heat behaviour of the underlying semi-Riemannian metric are the same along the flow of this 2-CVF.

Theorem 5.2. *Let (M, g) be a semi-Riemannian manifold and X be a 2-conformal vector field with conformal factor σ, i.e., $L_X L_X g = 2\sigma g$. Then (M, g) is a hyperbolic Ricci soliton with potential field X if and only if (M, g) is an h-almost Ricci soliton with potential field X, where $h = 2\lambda$ and the soliton function is $\tilde{\lambda} = \mu - 2\sigma$.*

Proof. By Definition 5.1, (M, g) is an HRS with potential field X if

$$L_X L_X g + \lambda L_X g + \mathrm{Ric} = \mu g.$$

Since X is a 2-CVF, we substitute $L_X L_X g = 2\sigma g$ into the HRS equation to obtain

$$2\sigma g + \lambda L_X g + \mathrm{Ric} = \mu g.$$

Rearranging the terms, we get

$$\lambda L_X g + \mathrm{Ric} = (\mu - 2\sigma) g.$$

Multiplying the equation by 2 yields

$$2\lambda L_X g + 2\mathrm{Ric} = 2(\mu - 2\sigma) g.$$

This is precisely the defining equation of an h-almost Ricci soliton $hL_X g + 2\mathrm{Ric} = 2\tilde{\lambda} g$ with $h = 2\lambda$ and soliton function $\tilde{\lambda} = \mu - 2\sigma$. Conversely, if (M, g) is an h-almost Ricci soliton with $h = 2\lambda$ and X being a 2-CVF, tracing back the steps yields the HRS equation. This completes the proof. □

References

[1] D. E. Allison, *Geodesic completeness in static spacetimes*, Geom. Dedicata, **26** (1988), 85–97.

[2] D. E. Allison, *Energy conditions in standard static space-times*, Gen. Relativ. Gravit., **20**(2) (1998), 115–122.

[3] D. E. Allison and B. Unal, *Geodesic structure of standard static space-times*, J. Geom. Phys., **46**(2) (2003), 193–200.

[4] A. L. Besse, *Einstein Manifolds*, Classics in Mathematics, Springer-Verlag, Berlin, 2008.

[5] R. L. Bishop and B. O'Neill, *Manifolds of negative curvature*, Trans. Amer. Math. Soc., **145** (1969), 1–49.

[6] D. Blair, *Almost contact manifolds with Killing structure tensors*, Pacific J. Math., **39**(2) (1971), 285–292.

[7] S. Deshmukh, *Conformal vector fields and eigenvectors of Laplacian operator*, Math. Phys. Anal. Geom., **15** (2012), 163–172.

[8] S. Deshmukh and F. R. Al-Solamy, *Conformal vector fields on a Riemannian manifold*, Balkan J. Geom. Appl., **19**(2) (2014), 86–93.

[9] Gh. Fasihi-Ramandi, S. Azami and M. A. Choudhary, *On hyperbolic Ricci solitons*, Chaos Solitons Fractals, **193** (2025), 95–116.

[10] W. Dai, W. Kong and D. Liu, *Hyperbolic geometric flow (I): Short-time existence and nonlinear stability*, Pure Appl. Math. Q., **6**(2) (2010).

[11] R. S. Hamilton, *Three-manifolds with positive Ricci curvature*, J. Differential Geom., **17**(2) (1982), 255–306.

[12] R. S. Hamilton, *The Ricci flow on surfaces*, Contemp. Math., **71** (1988), 237–261.

[13] W. Kuhnel and H. Rademacher, *Conformal vector fields on pseudo-Riemannian spaces*, J. Geom. Appl., **7** (1997), 237–250.

[14] M. Sanchez, *On the geometry of generalized Robertson–Walker spacetimes: Geodesics*, Gen. Relativ. Gravit., **30**(6) (1998), 915–932.

[15] M. Sanchez, *On the geometry of generalized Robertson–Walker spacetimes: Curvature and Killing fields*, J. Geom. Phys., **31**(1) (1999), 1–15.

[16] B. O'Neill, *Semi-Riemannian Geometry with Applications to Relativity*, Academic Press, London, 1983.

[17] T. Oprea, *2-Killing vector fields on Riemannian manifolds*, Balkan J. Geom. Appl., **13**(1) (2008), 87–92.

[18] S. Shenawy and B. Unal, *2-Killing vector fields on warped product manifolds*, Int. J. Math., **26**(9) (2015), 1550065.