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Research Paper

SOME RESULTS ON BAYES ESTIMATION UNDER LINEX LOSS FUNCTION

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Abstract. In this paper, we introduced straightforward formulas for the Bayes risk linked to the Linex loss function, which we then applied to estimate parameters of the normal, Poisson, and fractional Weibull distributions. We aimed to investigate the development of a linear Bayes estimator using the Linex loss function and successfully derived it for the normal and Poisson scenarios. We also demonstrated the process of creating empirical Bayes estimates using Linex loss and applied it to observed frequencies $f_n(x)$ produced by the Poisson-gamma model.

Key Words: Bayes Risk, Linex Loss Function, Linear Bayes Estimation, Empirical Bayese Estimation
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1. Introduction

The loss function $l(\theta, \hat{\theta})$ provides a measure of financial consequences arising from a wrong estimate $\hat{\theta}$ of the unknown θ. The selection of the suitable loss function is based solely on financial factors and is unrelated to the estimation method to be employed. The Bayes approach

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allows the economic considerations formulated through the loss (or utility) function to be used in a rational manner. A combination of the chosen loss function and the updated information, which is the posterior density function in Bayesian terms, actually yields the posterior expectation of the loss, and the value $\hat{\theta}$ that minimizes the expected loss is the Bayes estimation of θ. The point estimate is, of course, relative to the loss function. Several authors, for example [5, 7, 8] and [9] have pointed out that in some situations the use of symmetric loss functions may be inappropriate. In dam constructions, under estimation of peak water level is more serious than over estimation. However, in estimation of reliability function or average failure time over-estimation is more serious than under estimation. The 1986 disaster of the space shuttle challenger is attributed to over-estimation of the reliability of key space shuttle components. Roberts in [10] suggests the use of asymmetric loss function and, therefore, conservative estimate are generally used to avoid legal action. *Linex loss function* is a asymmetric loss functions. Linex means linear exponential loss function which used in the analysis of statistical estimation and prediction problem which rises exponentially on one side of zero and almost linearly on the other side of zero. It is used in both overestimation and underestimation problems. This loss function is a popular choice in statistics and machine learning for its ability to handle asymmetric errors. It is a versatile tool for modeling data and making predictions with a focus on minimizing the impact of outliers. This loss function strikes a balance between the mean absolute error and the mean squared error, offering a flexible approach that can be tailored to specific modeling needs.

Recently authors in [15] proposed the Linex loss function to determine optimum process parameters for the product quality. In this paper, we obtain some results for the Bayes risk based on the Linex loss function and apply them to estimate parameters of the Poisson, normal,and fractional Weibull distributions in section 3. An exploration in section 4 allows for the creation of a linear Bayes estimator utilizing the Linex loss function. We then effectively derived it for the normal and Poisson distributions. In section 5, we demonstrate a process of creating empirical Bayes estimates using Linex loss and applying it to observed frequencies $f_n(x)$ produced by the Poisson-gamma model.

2. Literature review

A very useful asymmetric loss function known as Linex loss function is a loss function used in machine learning and statistics. It is similar to the squared error loss function, but it has the advantage of being less sensitive to outliers. It is often used in regression problems where outliers are present in the data. The Linex loss function proposed by [8] which rises exponentially on one side of zero and almost linear on the other side of zero. The proposed Linex loss function (LLF) is

$$l(\Delta) = be^{a\Delta} - c\Delta - b, \quad a \neq 0,\, b > 0,$$

where $\Delta = \theta - \hat{\theta}$. It is clear that $l(0) = 0$ and for minimum to occur at $\Delta = 0$, one must have $ab = c$. Thus $l(\Delta)$ can be written as

$$l(\Delta) = b(e^{a\Delta} - a\Delta - 1), \quad a \neq 0,\, b > 0. \tag{2.1}$$

There are two parameters, a and b, involved in (2.1) with b serving to scale the loss function and a serving to determine its shape. The sign of the shape parameter a reflects the direction of the asymmetry ($a > 0$ is over-estimation is more serious than under-estimation, and vice-versa) and the magnitude of a reflects the degree of asymmetry. Authors in [4] derived the general expressions for the Bayes estimate $\hat{\theta}$ under LLF relative to $\Delta = \theta - \hat{\theta}$, which involves the moment generating function of the posterior density and one may require numerical methods to evaluate it. They also found Linex estimates in closed form for estimation problems concerning normal models and investigated risk properties of the obtained estimators. Authors in [14] obtained the Linex estimator for the mean of a Poisson distribution and investigated the risk properties of the estimator relative to those of linear transformation of the sample mean. In a series of papers, Basu and collaborators (see [13]) studied various problems concerning reliability function by considering a modified form of the Linex loss. Authors in [3] used asymmetric modified LLF to obtain an estimate of the scale parameters in exponential and normal families. They obtained best scale-invariant estimators of the parameters and conjectured their admissibility in the absence of nuisance parameters. In a study, [6] used LLF in conjunction with Amsters modified Bayes look-ahead rule for determining sample size in a one-armed phase II clinical trial. Linex loss has been found useful for location parameter estimation, however, it is found to be inconvenient for the estimation of scale parameter and other quantities. Authors in

[2] used the general Entropy loss in place of modified Linex loss for estimation of the scale parameter and other quantities.

3. Bayes Estimation

Let $X = x$ be an observation from a population with probability density function (or probability mass function) $f(x \mid \theta)$ with a parameter θ of the parameter space Θ. Let $p(x \mid \theta)$ be the posterior density (or mass) function with respect to the prior $p(\theta)$. Authors in [4] showed that the Bayes estimate $\hat{\theta}_L$ under LLF is

$$\hat{\theta}_L = -\frac{1}{a} \ln[E(e^{-a\theta} \mid X)],$$

Provided, of course, the posterior expectation exists and is finite. This involves evaluation of the moment generating function of the posterior density function. It is easy to show that as the shape parameter a of LLF tends to zero, $\hat{\theta}_L$ tends to the Bayes estimate $\hat{\theta}_S$ under the squared error loss as

$$\begin{aligned} \lim_{a\to 0}(-\frac{1}{a}) \ln[E(e^{-a\theta} \mid X)] &= \lim_{a\to 0} \Big(E(\theta e^{-a\theta} \mid X)/E(e^{-a\theta} \mid X)\Big) \\ &= E(\theta \mid X) = \hat{\theta}_S. \end{aligned}$$

After the value X has been observed and the Bayes Linex estimate $\hat{\theta}_L$ has been chosen, the risk is

$$R(\hat{\theta}_L, \theta) = bE^f(e^{a\Delta} - a\Delta - 1),$$

where E^f is the expectation taken with respect to $f(x \mid \theta)$ and $\Delta = \theta - \hat{\theta}$. The Bayes risk of $\hat{\theta}_L$ is obtained by

$$r(\pi, \hat{\theta}_L) = E^\pi \left(R(\hat{\theta}_L, \theta)\right),$$

where E^π is the expectation taken with respect to the prior $\pi(\theta)$.

Lemma 3.1. *Bayes risk associated with $\hat{\theta}_L$ is given by*

$$r(\pi, \hat{\theta}_L) = abE^\pi[E^f(\theta - \hat{\theta}_L)].$$

Proof. Denote

$$m(x) = \int_\Theta f(x \mid \theta)p(\theta)d\theta.$$

Since

$$\begin{aligned} E^{\pi}E^{f}\left(e^{a(\hat{\theta}_L-\theta)}\right) &= E^{m}E^{p}\left(e^{a(\hat{\theta}_L-\theta)}\right) \\ &= E^{m}e^{a\hat{\theta}_L}E^{p}(e^{-a\theta}) \\ &= E^{m}e^{a\hat{\theta}_L}e^{-a\hat{\theta}_L} = 1, \end{aligned}$$

where E^m is the expectation with respect to the marginal density $m(x)$ of X, we have

$$r(\pi,\hat{\theta}_L) = bE^{\pi}E^{f}\left(e^{a(\hat{\theta}_L-\theta)} - a(\hat{\theta}_L-\theta) - 1\right) = abE^{\pi}\left(E^{f}(\theta-\hat{\theta}_L)\right).$$

□

Theorem 3.2. *Bayes risk associated with $\hat{\theta}_L$ is given by*

$$r(\pi,\hat{\theta}_L) = abE^{m}(\hat{\theta}_S-\hat{\theta}_L),$$

where $\hat{\theta}_S = E^p(\theta)$ and E^p is the expectation taken with respect to $p(\theta \mid X)$.

Proof. By lemma 3.1, the proof is obvious, since

$$E^{\pi}E^{f}g(X,\theta) = E^{m}E^{p}\left(g(X,\theta)\right).$$

□

Remark 3.3. It is easy to see that

$$a(\hat{\theta}_S-\hat{\theta}_L) = \ln\left[E(e^{-a\theta} \mid X)/e^{-aE(\theta|X)}\right].$$

Then by Jensen inequality for $a \geq 0$, we have $\hat{\theta}_S \geq \hat{\theta}_L \geq 0$ and for $a \leq 0$, $\hat{\theta}_L \geq \hat{\theta}_S \geq 0$.

Now, we apply the recent results to estimate parameters of some special distributions such as the normal and Poisson.

Example 3.4. Let $X_1, X_2, , X_n$ be a random sample from $N(\theta, 1)$ and suppose $N(0,1)$ is the prior density for the unknown mean θ. We know that the posterior density of θ is $N(\frac{n\bar{x}}{n+1}, \frac{1}{n+1})$. Thus $\hat{\theta}_S = \frac{n\bar{x}}{n+1}$ and $\hat{\theta}_L = \frac{n\bar{x}}{n+1} - \frac{a}{2(n+1)}$ and $\hat{\theta}_L$ is a conservative estimate of θ in comparison to $\hat{\theta}_S$ for $a \geq 0$.
On applying theorem 3.2, the Bayes risk under LLF is easily computed

$$r(\pi,\hat{\theta}_L) = abE^{m}\left(\frac{n\bar{X}}{n+1} - \frac{n\bar{X}}{n+1} + \frac{a}{2(n+1)}\right) = \frac{a^2 b}{2(n+1)}.$$

The Bayes risk under square error loss function (SELF) is $r(\pi, \hat{\theta}_S) = 1/(n+1)$. Note that $r(\pi, \hat{\theta}_S) = r(\pi, \hat{\theta}_L)$, if $b = \frac{2}{a^2}, a \neq 0$.

Example 3.5. Suppose that $X_1, X_2, , X_n$ is a random sample from a Poisson distribution with unknown mean θ and let the prior density of θ be $\Gamma(\alpha, \beta), \alpha > 0, \beta > 0$. The posterior density of θ, given $X_i = x_i, i = 1, 2, ..., n$, is also gamma but with revised parameters $\alpha + \Sigma_{i=1}^{n} x_i$ and $\beta + n$. The Bayes estimate under SELF is $\hat{\theta}_S = \frac{\alpha + \Sigma_{i=1}^{n} x_i}{\beta + n}$. Sadooghi-Alvandi in [14] gives the Linex Bayes estimate for θ as

$$\hat{\theta}_L = \frac{\alpha + \Sigma_{i=1}^{n} x_i}{\alpha} \ln[1 + \frac{\alpha}{\beta + n}], \quad a + \beta + n > 0.$$

The Bayes risk under LLF is, once again, easily obtained by Theorem 3.2, as

$$\begin{aligned} r(\pi, \hat{\theta}_L) &= abE^m(\hat{\theta}_S - \hat{\theta}_L) \\ &= abE^m \left(\frac{\alpha + \Sigma_{i=1}^{n} X_i}{n + \beta} - \frac{\alpha + \Sigma_{i=1}^{n} X_i}{\alpha} \ln[1 + \frac{\alpha}{\beta + n}] \right) && (3.1) \\ &= \frac{b\alpha}{\beta} \left(a - (\beta + n) \ln[1 + \frac{a}{n + \beta}] \right), && (3.2) \end{aligned}$$

since $E(X) = E^{\pi} E^{f}(X \mid \theta) = \alpha/\beta$, which is same as Eq. (2.4) for Bayes risk from [14].

Theorem 3.6. *Let $l(\Delta) = b(e^{a\Delta} - a\Delta - 1)$ and $\Delta = \theta - \hat{\theta}$. Then, no Bayes estimator can be unbiased unless its Bayes risk is zero.*

Proof. Suppose that for a proper prior π, the Bayes estimator $\delta_\pi(x)$ is unbiased,

$$E^f(\delta_\pi(X)) = \theta, \quad \forall \theta.$$

This implies that $r(\pi, \delta_\pi) = 0$. The Bayes risk of $\delta_\pi(x)$ can be calculated as repeated expectation in two ways

$$\begin{aligned} r(\pi, \delta_\pi) =& E^{\pi} E^{f} \left(b(e^{a(\delta_\pi - \theta)} - a(\delta_\pi - \theta) - 1) \right) \\ =& E^{m} E^{p} \left(b(e^{a(\delta_\pi - \theta)} - a(\delta_\pi - \theta) - 1) \right). \end{aligned}$$

Thus, conveniently choosing either EE^f or E^mE^p and using the properties of conditional expectation we have

$$\begin{aligned} r(\pi,\delta_\pi) &= bE^\pi E^f(e^{a\delta_\pi}.e^{-a\theta}) - abE^\pi E^f(\delta_\pi) + abE^\pi E^f(\theta) - b \\ &= bE^m E^p(e^{a\delta_\pi}.e^{-a\theta}) - abE^\pi E^f(\delta_\pi) + abE^\pi(\theta) - b \\ &= bE^m(e^{a\delta_\pi}E^p(e^{-a\theta}) - abE^\pi E^f(\delta_\pi) + abE^\pi(\theta) - b \\ &= bE^m(e^{a\delta_\pi}.e^{-a\delta_\pi}) - abE^\pi(\theta) + abE^\pi(\theta) - b. \end{aligned}$$

□

Now, we present an identity relating the Bayes risk to bias, which illustrates that a small bias can help achieve a small Bayes risk. Let $X \sim f(x \mid \theta)$ and $\theta \sim \pi(\theta)$. We know that the Bayes estimator under linex loss function is

$$\delta_\pi(x) = -\frac{1}{a}\ln[E^p(e^{-a\theta} \mid X)].$$

The Bayes risk of δ_π can be written as

$$r(\pi,\delta_\pi) = -ab\int b(\theta)\pi(\theta)d\theta,$$

where $b(\theta) = E^f(\delta_\pi(X)) - \theta$ is the bias of δ_π. It is easy to see that

$$r(\pi,\delta_\pi) = b[E^\pi E^f(e^{a\delta_\pi}.e^{-a\theta}) - aE^\pi E^f(\delta_\pi) + aE^\pi E^f(\theta) - 1].$$

Since, $E^\pi e^{-a\theta}E^f(e^{a\delta_\pi}) = 1$, then we have

$$\begin{aligned} r(\pi,\delta_\pi) =& b[-aE^\pi E^f(\delta_\pi) + aE^\pi E^f(\theta)] \\ =& ab[E^\pi(\theta) - E^\pi E^f(\delta_\pi)] \\ =& -ab\int (E^f(\delta_\pi) - \theta)\pi(\theta)d\theta. \end{aligned}$$

Now, we study the Bayesian estimation of parameter θ of nabla discrete fractional Weibull distribution under linex loss function. A discrete analogue of Weibull distribution is the nabla discrete fractional Weibull distribution defined by the probability mass function ([12])

$$P(X=x) = \gamma\theta x^{\overline{\gamma-1}}(1-\theta)^{x^{\overline{\gamma}}-1}, \quad x = \mathbb{N}_1, \gamma > 0, 0 < \theta < 1,$$

where $\mathbb{N}_a = \{a, a+1, ...\}$ and $x^{\overline{\gamma}} = \Gamma(x+\gamma)/\Gamma(x)$ with $0^{\overline{\gamma}} = 0$.
The likelihood function of θ, in this case, is given by

$$L(\theta) \propto \theta^n(1-\theta)^{\Sigma x_i^{\overline{\gamma}} - n}.$$

We take a prior distribution given below

$$\pi(\theta) = \frac{\theta^{\alpha-1}(1-\theta)^{\beta-1}e^{-\mu\theta}}{B(\alpha,\beta)_1F_1(\beta,\alpha+\beta,-\mu)}, \quad 0<\theta<1, \alpha,\beta>0, \mu\in\mathbb{R},$$

with $B(\alpha,\beta)=\Gamma(\alpha)\Gamma(\beta)/\Gamma(\alpha+\beta)$, ${}_1F_1(a,b,c)=\sum_{k=0}^{\infty} a^{\overline{k}}\, c^k/b^k k!$ with the general form

$${}_pF_q(a_1,a_2,...,a_p,b_1,b_2,...,b_q,c) = \sum_{k=0}^{\infty}\left(\prod_{i=1}^{p} a_i^{\overline{k}}\, c^k\right) / \left(\prod_{i=1}^{q} b_i^{\overline{k}}\, k\right)$$

is called a generalized hypergeometric series. This prior density is known as Kummer-beta density and denoted by $KB(\alpha,\beta,\mu)$. The posterior density of θ, corresponding to $\pi(\theta)$, is given by

$$\begin{aligned}\pi(\theta|x) \propto& \theta^{n+\alpha-1}(1-)^{\Sigma x_i^{\overline{\gamma}}-n+\beta-1}e^{-\mu\theta},\\ &\Rightarrow\ \theta \mid x \sim KB(n+\alpha, \Sigma x_i^{\overline{\gamma}}-n+\beta,\mu).\end{aligned}$$

Under LLF, the Bayes estimate θ, corresponding to posterior density $\pi(\theta|x)$, is given by

$$\begin{aligned}\hat{\theta}_L =& \ln_1 F_1^{a^{-1}}\left(\Sigma x_i^{\overline{\gamma}}-n+\beta, \Sigma x_i^{\overline{\gamma}}+\alpha+\beta,-\mu\right)\\ &-\ln_1 F_1^{a^{-1}}\left(\Sigma x_i^{\overline{\gamma}}-n+\beta, \Sigma x_i^{\overline{\gamma}}+\alpha+\beta,-\mu-a\right).\end{aligned}$$

Since, it is easy to see that

$$E^p(e^{-a\theta}) = \frac{{}_1F_1\left(\Sigma x_i^{\overline{\gamma}}-n+\beta, \Sigma x_i^{\overline{\gamma}}+\alpha+\beta,-\mu-a\right)}{{}_1F_1\left(\Sigma x_i^{\overline{\gamma}}-n+\beta, \Sigma x_i^{\overline{\gamma}}+\alpha+\beta,-\mu\right)}.$$

A discrete analogue of Weibull distribution is the delta discrete fractional Weibull distribution defined by the probability mass function ([12])

$$P(X=x) = \frac{\gamma\theta x^{\underline{\gamma-1}}}{(1+\theta)^{x^{\underline{\gamma}}+1}}, \quad x=\mathbb{N}_{\gamma-1}, \gamma>0, \theta<0,$$

where $x^{\underline{\gamma}}=\Gamma(x+1)/\Gamma(x+1-\gamma)$. The likelihood function of θ, in this case, is given by $L(\theta)\propto\theta^n/(1+\theta)^{\Sigma x_i^{\underline{\gamma}}-n}$. We take a prior distribution given below

$$\pi(\theta) = \frac{\theta^{\alpha-1}}{B(\alpha,\beta)(1+\theta)^{\alpha+\beta}}, \quad \theta,\alpha,\beta>0.$$

This prior density is known as beta type II density and denoted by $B^{II}(\alpha, \beta)$. The posterior density of θ, corresponding to $\pi(\theta)$, is given by

$$\pi(\theta \mid X) \propto \theta^{n+\alpha-1}/(1+\theta)^{\Sigma x_i^{\gamma}+n+\alpha+\beta},$$
$$\Rightarrow \ \theta \mid X \sim B^{II}(n+\alpha, \Sigma x_i^{\gamma}+\beta).$$

$\pi(\theta)$ is a natural conjugate prior density. Note that $\pi(\theta)$ is a special case of inverted hypergeometric function type I density, which is given by ([1]). Under LLF, the Bayes estimate of θ, corresponding to posterior density $\pi(\theta \mid X)$, is given by

$$\hat{\theta}_L = \ln\left[\Gamma^{a^{-1}}(\Sigma x_i^{\gamma}+\beta)\right] - \ln\left[\Gamma^{a^{-1}}(\Sigma x_i^{\gamma}+n+\alpha+\beta)\psi^{a^{-1}}(n+\alpha, 1-(\Sigma x_i^{\gamma}+\beta), a)\right],$$

since, we have

$$\int_0^\infty e^{-px}x^{q-1}(1+ax)^{-\gamma}dx = a^{-q}\Gamma(q)\psi(q, q+1-\gamma, \frac{p}{a}).$$

4. Linear Bayes Estimation

A point estimator derived using the prior distribution to find the best estimator within a certain class of estimators can be called an approximate Bayes estimator if it is not the actual Bayes estimator. Meritz and Lwin in their excellent book on Empirical Bayes Estimation [16] consider a simple case in which there is just one observation x or X and the parameter value is θ to obtain linear Bayes estimator of θ under the squared error loss function. They find the class of linear Bayes estimators to be extremely useful in the study of empirical Bayes estimation (EBE). In this section we propose to explore the construction of a linear Bayes estimator under the Linex loss function and obtain it for the normal and Poisson cases. Let us consider the linear estimator of the form

$$\delta(w_0, w_1, x) = w_0 + w_1 x,$$

where w_0 and w_1 are chosen to minimize the risk $E^{\pi}E^{f}\left(l(\theta, \delta(w_0, w_1, x))\right)$. The terminology linear Bayes is explained by the form of $\delta(w_0, w_1, x)$ and the fact that that prior $\pi(\theta)$ plays a role in the determination of w_0 and w_1. The Bayes values $\hat{w}_0$ and $\hat{w}_1$ are easily obtained by differentiating the risk function with respect to $\hat{w}_0$ and $\hat{w}_1$, equating the derivatives to zero and solving the two equations for $\hat{w}_0$ and $\hat{w}_1$. In the case of SELF one gets linear simultaneous equations in $\hat{w}_0$ and $\hat{w}_1$ which have

immediate solution. The solution depends only on the first two moments of $m(x)$. However, we shall see that in the case of Linex loss, the two equations may be difficult to solve.
To obtain Bayes value, $\hat{w}_0$ and $\hat{w}_1$ under Linex loss, consider the derivatives of $E^\pi E^f l(\Delta)$ where $\Delta = \delta - \theta$, with respect to $\hat{w}_0$ and $\hat{w}_1$. Let

$$\delta = w_0 + w_1 x = \frac{1}{a}(u_0 + u_1 x), \quad a \neq 0,$$

then,

$$\frac{\partial}{\partial u_i} E^\pi E^f b\left(e^{a(\delta-\theta)} - a(\delta-\theta) - 1\right) = 0, \quad i = 1, 2,$$

gives

$$E^\pi E^f (e^{u_1 X - a\theta}) = e^{-u_0},$$

and

$$E^\pi E^f (X e^{u_1 X - a\theta}) = e^{-u_0} E^m(X),$$

or

$$\ln E^\pi \left(e^{-a\theta} M_{X|\theta}(u_1)\right) + u_0 = 0,$$

and

$$\ln E^\pi \left(e^{-a\theta} \frac{\partial}{\partial u_1} M_{X|\theta}(u_1)\right) = -u_0 + \ln E^m(X),$$

where $M_{X|\theta}(u_1)$ is the moment generating function for $f(x|\theta)$.

Example 4.1. Consider the normal example 3.4 of section 3. We want to obtain the linear Bayes estimator of θ under Linex loss. Since $m(x)$ is $(0, 2)$, the two equations (3.1)and (3.2) are to be solved for u_0 and u_1. We get the two equations as $\hat{w}_0$ and $\hat{w}_1$

$$E^\pi E^f (e^{u_1 X - a\theta}) = e^{-u_0},$$

and

$$E^\pi E^f (X e^{u_1 X - a\theta}) = 0,$$

or

$$u_0 = -\left(u_1^2 + (u_1 - a)^2\right)/2, \;\; u_1 = a/2,$$

thus

$$\delta(\hat{w}_0, \hat{w}_1, x) = -\frac{a}{4} + \frac{x}{2} = \frac{2x - a}{4},$$

with Bayes risk $r(\pi, \hat{\theta}_L) = \frac{a^2 b}{4}$.

In this example, the linear Bayes estimator of θ under Linex loss is exactly the actual Bayes estimator under Linex loss. However, if one uses other than conjugate prior for θ, linear Bayes estimator may be different than the actual Bayes estimator.

Example 4.2. Let us obtain the linear Bayes estimator for the Poisson mean θ under LLF. The marginal probability function of X with respect to $\Gamma(\alpha,\beta)$, where $\alpha > 0, \beta > 0$, prior for θ is the negative binomial distribution given by

$$m(x) = \frac{\Gamma(\alpha+x)}{\Gamma(x+1)\Gamma(\alpha)}(1-\frac{1}{\beta+1})^{\alpha}(\frac{1}{\beta+1})^{x}, \quad x = 0, 1, ,$$

with mean α/β and variance $\alpha(1+\beta)/\beta^2$.
In order to obtain the linear Bayes estimator, note that

$$\begin{aligned} E^f(e^{u_1 X}) &= e^{-\theta(1-e^{u_1})}, \\ E^{p(\theta)}E^f(e^{u_1 X - a\theta}) &= E^{p(\theta)}\left(e^{-\theta(1+a-e^{u_1})}\right) = \left(\beta/(1+a+\beta-e^{u_1})\right)^{\alpha}, \end{aligned}$$

and

$$E^f(Xe^{u_1 X}) = \frac{\Sigma_{x=1}^{\infty} e^{-\theta}(\theta e^{u_1})^x}{(x-1)!} = \theta e^{u_1} e^{-\theta(1-e^{u_1})},$$

so that,

$$\begin{aligned} E^{p(\theta)}E^f(Xe^{u_1 X - a\theta}) &= e^{u_1}E^{p(\theta)}\left(\theta e^{-(1-e^{u_1})\theta-\theta}\right) \\ &= \frac{\alpha}{\beta}e^{u_1}\left(\beta/(1+a+\beta-e^{u_1})\right)^{1+\alpha}. \end{aligned}$$

Then Eq. (3.2) reduced to the pair of equations

$$\alpha \ln\left[\beta/(1+a+\beta-e^{u_1})\right] + u_0 = 0,$$

and

$$\ln\left[\frac{\alpha}{\beta}e^{u_1}\left(\frac{\beta}{1+a+\beta-e^{u_1}}\right)^{1+\alpha}\right] - \left(\ln[\frac{\alpha}{\beta}] - u_0\right) = 0.$$

Thus $\hat{u}_0 = \alpha\ln[1+\frac{a}{1+\beta}]$, $\hat{u}_1 = \alpha\ln[1+\frac{a}{1+\beta}]$ and the linear Bayes estimate of θ is

$$\delta(x) = \frac{1}{a}(\alpha+x)\ln[1+\frac{a}{1+\beta}],$$

which is the same as $\hat{\theta}_L$ for $n = 1$.

Remark 4.3. In both cases, the Bayes estimate is a linear function of X, so the linear Bayes estimate should match the actual Bayes estimate. However, in general, the linear Bayes estimate may not be identical to the actual Bayes estimate.

5. Linear Empirical Bayes Estimation for Poisson Distribution

In this section, we illustrate the method of constructing empirical Bayes estimate (EBE) under Linex loss. We use the data given in Table 3.2 from [16]. Table 1 shows the observed frequencies $f_n(x)$ generated by the Poisson distribution mixed by a $\Gamma(\alpha, \beta)$ prior, where $\alpha = 10$ and $\beta = 2$. Here

$$p(\theta) = \frac{\beta^{\alpha}}{\Gamma(\alpha)}\theta^{\alpha-1}e^{-\beta\theta}, \quad \theta \geq 0, \alpha > 0, \beta > 0.$$

For these data $\bar{x} = 5.0, s^2 = 9.12$ and the estimates of α and β obtained by the method of moments are $\alpha = 6.068$ and $\beta = 1.214$ (see [16], page 84). The linear EBE under SELF is found to be

$$\delta_S(x) = 2.741 + 0.452x,$$

and the linear EBE under LLF, given in (13), becomes

$$\delta_L(x) = \frac{1}{a}(6.068 + x)\ln[1 + \frac{a}{2.214}].$$

Table 1 gives the comparative values of $\delta_S(x)$ and $\delta_L(x)$ for $a = -2, -0.5, 0.5$ and 2.0.

We note that,

(I) $\delta_L(x)$ (x) decrease as the value of the shape parameter a increase from $a = -2$ to $a = 2$, for all x.

(II) $\delta_S(x)$ is close to $\delta_L(x)$ for $\mid a \mid \leq 0.5$.

(III) Rate of increase in $\delta_L(x)$ value decreases with an increase in the value of shape parameter a.

Furthermore, Bayes risk of the linear EBE under SELF is $\alpha/\beta(\beta+1)$ and corresponding Bayes risk under LLF is given by Eq. (3.2). Using the estimates of α and β (obtained by the method of moments) one finds

$$\hat{r}(\pi, \hat{\theta}_S) = 2.258,$$

and $\hat{r}(\pi, \hat{\theta}_S)$ are tabulated in Table 2 for $a = -2, -0.5, 0.5$ and 2.0. Table 2 gives the Bayes risk under LLF when $b = 1$. We see that the Bayes risk behaves in an asymmetric fashion with respect to a. Bayes risk when $a = -2$ is much larger than the Bayes risk when $a = 2$.

TABLE 1. Observed frequencies $f_n(x)$ generated by the Poisson-gamma model when $\alpha = 10, \beta = 2$ Columns headed by $(a), (b), (c)$ and (d) are $\hat{\delta}_L$ for $a = -2, -0.5, 0.5$ and 2, respectively

x	$f_n(x)$	$\hat{\hat{\delta}}_S$	(a)	(b)	(c)	(d)
0	-	2.741	7.089	3.105	2.471	1.954
1	3	3.193	8.257	3.618	2.878	2.275
2	8	3.645	9.426	4.130	3.286	2.596
3	10	4.097	10.594	4.642	3.693	2.918
4	2	4.549	11.762	5.154	4.100	3.240
5	11	5.001	12.931	5.666	4.507	3.562
6	4	5.453	14.099	6.178	4.915	3.884
7	4	5.905	15.267	6.690	5.322	4.205
8	-	6.357	16.436	7.202	5.729	4.527
9	1	6.809	17.604	7.714	6.136	4.849
10	2	7.261	18.772	8.226	6.544	5.171
11	4	7.713	19.940	8.738	6.951	5.493
12	-	7.165	21.109	9.250	7.358	5.814
13	1	8.617	22.277	9.762	7.765	6.136
14	-	9.069	23.445	10.274	8.173	6.458
15	-	9.521	24.614	10.786	8.580	6.780
16	-	9.973	25.782	11.258	8.987	7.102

TABLE 2. Comparative values of estimated Bayes risk (based on the data given in Table 1) under Linex loss function for $n = 1, b = 1$

a	-2	-0.5	0.5	2
Estimated Bayes risk	15.86	0.333	0.246	2.874

6. CONCLUSION

Processes are described for constructing linear Bayes estimators and empirical Bayes estimators using the Linex loss function. Processes are described for constructing linear Bayes estimators and empirical Bayes estimators using the Linex loss function.The construction of these estimators involves minimizing the Linex loss function with respect to the unknown parameters. Linear Bayes estimators are obtained by assuming a prior distribution for the parameters and deriving the posterior distribution using Bayes' theorem. Empirical Bayes estimators, on the other hand, use the data itself to estimate the prior distribution, leading to a more data-driven approach. Both methods aim to find estimators that balance the bias-variance trade-off by incorporating prior information and empirical data. Moreover, explicit formulas for Bayes risk under the Linex loss function are presented. Special distributions are applied to demonstrate the superiority of the results.

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