Journal of Hyperstructures 13 *(1) (2024), 109-123.*
https://doi.org/10.22098/jhs.2024.14825.1011
Research Paper

APPROACH TO FUZZY DIFFERENTIAL EQUATIONS IN INTUITIONISTIC FUZZY METRIC SPACES USING GENERALIZED CONTRACTION THEOREMS

Ram Milan Singh, Deepak Singh and Rahul Gourh

Abstract. Approach to fuzzy differential equations in fuzzy metrics using generalized contraction theorems One of the most fascinating fields of mathematical inquiry is fixed point.In this direction, by utilizing "the triangular property of fuzzy metric ", we investigate certain special common fixed point solutions for a pair of self mappings without continuity on fuzzy metric spaces under the generalized contra ction requirements.We also provide generalized Ciri's contraction theorems and weak contraction theorems. The findings are corroborated by pertinent cases. Additionally, we provide a workable application of the fuzzy differential equation s to guarantee the presence of a singular common solution to support our primary research.

Key Words: Intuitionistic Fuzzy norm, t-norm, fuzzy differential equations, Intuitionistic fuzzy differential equations, contraction conditions, fuzzy metric space, Intuitionistic fuzzy metric space.
2010 Mathematics Subject Classification: Primary: 54A40; Secondary: 03E72

1. Introduction

The term "fuzzy set" was first used by Zadeh [1] in 1965 and is defined as "a set constructed from a function having a domain is a nonempty set W and range in [0,1]" (i.e., if G: W [0,1], then the set constructed from the mapping G is termed a fuzzy set). The theory of fuzzy

Received: 18 March 2024, Accepted: 15 May 2024. Communicated by Hoger Ghahramani;
∗Ram Milan Singh; E-mail: rammilansinghlig@gmail.com.

sets has since undergone substantial development and investigation in a variety of directions with several applications. The idea of fuzzy metric spaces (FM spaces) was first developed by Kramosil and Michalek [2] using the fuzzy set concept and some additional, derived notions fro m the one in order. They demonstrated that, in some situations, the FM ideas and the statistical metric space are identical. Following that, George and Veeramani [3] provided the modified version of the F M space, demonstrating that every metric produces an FM. They established some fundamental truths including the Baire theorem for FM spaces.

By utilizing the idea of Kramosil and Michalek [2], Grabiec [4] proved two fixed p oint theorems of "Banach and Edelstein contraction mapping theorems on compl ete and compact FM spaces, respectively" in 1988. Fixed point theorems for a fa mily of mappings on FM spaces were proven by Bari and Vetro [6]. While certain invariant approximation findings for fuzzy non expansive mappings constructed on FM spaces were proven by Beg et al. [7]. Additionally, they found a required condition for the set of all tbest approximations that contained a fixed point of the arbitrary mappings and created the strictly convex fuzzy normed space.

In this study, we use "the triangular property of fuzzy metric" to provide some no vel common fixed point theorems for a pair of self mappings on FM spaces without continuity. Additionally, we offer an extended Ciri'c-contraction theorem on FM space as well as weak contraction. To further complement our study, we also show an application of fuzzy differenti al equations.

2. Preliminaries

In this section, some definitions and results are collected which are used in this paper.

Definition 2.1 . A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is called a t-norm if it satisfies the following conditions:

(a) $*$ is commutative and associative;
(b) $*$ is continuous;
(c) $\mathrm{a} * 1 = \mathrm{a}$ for all $\mathrm{a} \in [0,1]$;
(d) $\mathrm{a} * \mathrm{b} \leq \mathrm{c} * \mathrm{d}$ whenever $\mathrm{a} \leq \mathrm{c}$ and $\mathrm{b} \leq \mathrm{d}$, for each $\mathrm{a,b,c,d} \in [0,1]$.

If $*$ is continuous, then it is called continuous t-norm.
The following are examples of some t-norms.
(i) Standard intersection: $a * b = \min\{a, b\}$.
(ii) Algebraic product: $a * b = ab$.
(iii) Bounded difference: $a * b = \max\{0, a + b - 1\}$.

Definition 2.2 A binary operation $\diamond : [0,1] \times [0,1] \to [0,1]$ is a continuous tconorm if it satisfies the following conditions:

(a) $\diamond$ is commutative and associative;
(b) is continuous;
(c) $a - 0 =$ a for all $\mathrm{a} \in [0,1]$;
(d) a $\diamond b \leq \mathrm{c} \diamond$ d whenever a $\leq$ c and b $\leq$ d, for each a, b, c, d $\in [0,1]$:

If $*$ is continuous, then it is called continuous t-norm.
The following are examples of some t-norms.
(i) Standard intersection: $ab = \max\{a, b\}$.
(ii) Algebraic product: $a \vee b = ab$.
(iii) Bounded difference: $a \diamond b = \min\{0, a + b - 1\}$.

Definition 2.3 A three tuple $(X, M, *)$ is said to be a fuzzy metric space if X is an arbitrary set, $*$ a continuous t-norm and M a fuzzy set on $\mathrm{X}^2 \times [0, \infty)$ satisfying the following condition, for all $\mathrm{x,y,z} \in \mathrm{X}$ and $\mathrm{t,s} > 0$:

(a) $M(x, y, 0) = 0$
(b) $M(x, y, t) = 1$ for all $t > 0$ iff $x = y$,
(c) $M(x, y, t) = M(y, x, t)$,
(d) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$,
(e) $M(x, y,) : .[0, \infty) \rightarrow [0, 1]$ is left continuous,
(f) $\lim_{n\rightarrow\infty} M(x, y, t) = 1$.

Definition 2.4. A 5 -tuple (X, M, N, *,$\diamond$) is said to be an intuitionistic fuzzy metric space (shortly IFM-Space) if X is an arbitrary set, $*$ is a continuous t-norm, $\diamond$ is a continuous t-conorm and M, N are fuzzy sets on $X^2 \times [0, \infty)$ satisfying the following conditions:
(a) $M(x, y, t) + N(x, y, t) \leq 1$ for all $x, y \in X$ and $t > 0$;
(b) $M(x, y, 0) = 0$ for all $x, y \in X$; (c) $M(x, y, t) = 1$ for all $x, y \in X$ and $t > 0$ if and only if $x = y$,
(d) $M(x, y, t) = M(y, x, t)$ for all $x, y \in X$ and $t > 0$;
(e) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$ for all $x, y, z \in X$ and $s, t > 0$,
(f) $M(x, y,) : .[0, \infty) \rightarrow [0, 1]$ is left continuous for all $x, y \in X$
(g) $\lim_{n\rightarrow\infty} M(x, y, t) = 1$,
(h) $N(x, y, 0) = 1$ for all $x, y \in X$
(i) $N(x, y, t) = 0$ for all $x, y \in X$ and $t > 0$ if and only if $x = y$,
(j) $N(x, y, t) = N(y, x, t)$ for all $x, y \in X$ and $t > 0$,
(k) $N(x, y, t) \wedge N(y, z, s) \geq N(x, z, t + s)$ for all $x, y, z \in X$ and $s, t > 0$,
(l) $M(x, y,) : .[0, \infty) \rightarrow [0, 1]$ is right continuous for all $x, y \in X$
(m) $\lim_{n\rightarrow\infty} N(x, y, t) = 0$ for all $x, y \in X$;
Then (M, N) is called an intuitionistic fuzzy metric on X. The functions $M(x, y, t)$ and $N(x, y, t)$ denote the degree of nearness and degree of non nearness between x and y with respect to t, respectively.

Definition 2.5 Let (X, M, N, *,$\diamond$) be an intuitionistic fuzzy metric space. Then
(a) a sequence $\{x_n\}$ is said to be convergent x in X if for each $\epsilon > 0$ and $t > 0$, there exist $n_0 \in N$ such that $M(x_n, x, t) > 1 - \epsilon$ and $N(x_n, x, t) < 0 - \epsilon$ for all $n \geq n_0$.
(b) a sequence $\{x_n\}$ is said to be Cauchy if for each $\epsilon > 0$ and $t > 0$, there exist $n_0 \in N$ such that $M(x_n, x_m, t) > 1 - \epsilon$ and $N(x_n, x_m, t) < 0 - \epsilon$ for all $n, m \geq n_0$.
(c) An intuitionistic fuzzy metric space in which every Cauchy sequence is convergent is said to be complete.

Definition 2.6 A sequence {Si} of self maps on a complete intuitionistic fuzzy metric space $(X, M, N, *)$ is said to be intuitionistic mutually contractive if for $t > 0$ and $i \in \mathrm{N}$

$$M(S_i x, S_j y, t) \geq M\left(x, y, \frac{t}{p}\right) \text{ and } N(S_i x, S_j y, t) \leq N\left(x, y, \frac{t}{p}\right).$$

Where $x, y \in X, p \in (0, 1), i \neq j$ and $x \neq y$.

Definition 2.7 Let (X, M, N, *,$\diamond$) be a intuitionistic fuzzy normed linear space.
(i) A sequence $\{x_n\}$ is said to be convergent if there exists $x \in X$ such that $\lim_{n\rightarrow\infty} \mathrm{M}(x_n - x, t) = 1$ and $\lim_{n\rightarrow\infty} N(x_n - x, t) = 0$ for all $t > 0$. Then x is called the limit of the sequence $\{x_n\}$ and denoted by $\lim_{n\rightarrow\infty} x_n$.
(ii) A sequence $\{x_n\}$ in a intuitionistic fuzzy normed linear space (X, N) is said to be Cauchy if $\lim_{n\rightarrow\infty} \mathrm{M}(x_{n+p} - x_n, t) = 1$ and $\lim_{n\rightarrow\infty} N(x_{n+p} - x_n, t) = 0$ for all $t > 0$ and $p = 1, 2, \ldots$
(iii) $A \subseteq X$ is said to be closed if for any sequence $\{x_n\}$ in A converges to $x \in A$.

(iv) $A \subseteq X$ is said to be the closure of A, denoted by $\bar{A}$ if for any $x \in \bar{A}$, if there is a sequence $\{x_n\} \subseteq A$ such that $\{x_n\}$ converges to x.
(v) $A \subseteq X$ is said to be compact if any sequence $\{x_n\} \subseteq A$ has a subsequence converging to an element of A.

Definition 2.8 Let (X, M, N, *,$\diamond$) be a intuitionistic fuzzy normed linear space.
(i) A set $B(x, \alpha, t), 0 < \alpha < 1$ is defined as
$B(x, \alpha, t) = \{y : M(x - y, t) > 1 - \alpha\}$. and $B(x, \alpha, t) = \{y : N(x - y, t) < 1 - \alpha\}$.
(ii) $\tau = \{G \subseteq X : x \in G, \exists t > 0, 0 < \alpha < 1$ such that $B(x, \alpha, t) \subset G\}$ is a topology on (X, M, N).
(iii) Members of τ are called open sets in (X, M, N).

Definition 2.9. A subset B of a fuzzy normed linear space (X, M, N) is said to be intuitionistic fuzzy bounded if for each $r, 0 < r < 1, \exists t > 0$ such that

$$M(x, t) > 1 - r \text{ and } N(x, t) < 1 - r \text{ for all } x \in B.$$

Lemma 2.6 Let (X, M, N) be a intuitionistic fuzzy normed linear space and $M(x,.)$ $and N(x, \cdot)(x \neq 0)$. If the set $A = \{x : M(x, 1) > 0$ and $\mathrm{N(x, 1)} < 0\}$ is compact, then X is finite dimensional.

3. **Main Result**

3.1. **Generalized iri-contraction results on IFM spaces.** In this section, we define a generalized iri type of Intuitionistic fuzzy contraction on IFM spaces and present a unique common fixed point theorem for a pair of selfmappings on a complete IFM space.

Definition 3.1. Let $(X, \mathrm{M}, *)$ be an FM space. A self-mapping $F_1 : X \to X$ is said to be a generalized iri type fuzzy-contraction if $\exists \alpha \in (0, 1)$ such that

$$\frac{1}{M\left(F_1 x, F_1 y, t\right)} - 1 \leq \alpha \max \left\{ \begin{array}{c} \left(\frac{1}{\mathrm{M}(x,y,t)} - 1\right), \\ \left(\frac{1}{M(x,F_1 x,t)} - 1\right), \left(\frac{1}{\mathrm{M}(y,F_1 y,t)} - 1\right), \\ \left(\frac{1}{\mathrm{M}(y,F_1 x,t)} - 1\right), \left(\frac{1}{\mathrm{M}(x,F_1 y,t)} - 1\right), \\ \frac{1}{2}\left(\frac{1}{\mathrm{M}(x,F_1 x,t)} - 1 + \frac{1}{\mathrm{M}(y,F_1 y,t)} - 1\right), \\ \frac{1}{2}\left(\frac{1}{\mathrm{M}(y,F_1 x,t)} - 1 + \frac{1}{\mathrm{M}(X,F_1 y,t)} - 1\right) \end{array} \right\}$$

$\forall x, y \in X$ and $t > 0$.

In the following, we present a more generalized iri type fuzzy contraction result for a pair of self-mappings to prove that a pair of self-mappings on a complete FM space have a unique common fixed point.

Definition 4.2.. Let $(X, M,\ \mathrm{N}, *, \diamond)$ be a complete IFM space in which M is triangular and N is co-triangular and a pair of self mappings $f_1, f_2 : X \to X$ satisfies,

$$\frac{1}{M\left(F_1 x, F_1 y, t\right)} - 1 \leq \alpha \max \left\{ \begin{array}{c} \left(\frac{1}{\mathrm{M}(x,y,t)} - 1\right), \\ \left(\frac{1}{M(x,F_1 x,t)} - 1\right), \left(\frac{1}{\mathrm{M}(y,F_1 y,t)} - 1\right), \\ \left(\frac{1}{\mathrm{M}(y,F_1 x,t)} - 1\right), \left(\frac{1}{\mathrm{M}(x,F_1 y,t)} - 1\right), \\ \frac{1}{2}\left(\frac{1}{\mathrm{M}(x,F_1 x,t)} - 1 + \frac{1}{\mathrm{M}(y,F_1 y,t)} - 1\right), \\ \frac{1}{2}\left(\frac{1}{\mathrm{M}(y,F_1 x,t)} - 1 + \frac{1}{\mathrm{M}(X,F_1 y,t)} - 1\right) \end{array} \right\}$$

and

$$+\beta \max \left\{ \begin{array}{c} = \alpha \left(\frac{1}{\mathrm{M}\left(x_{2m+1}, \kappa, t\right)} - 1 \right) \\ \left(\frac{1}{\mathrm{M}\left(x_{2m+1}, \kappa, t\right)} - 1 \right), \left(\frac{1}{\mathrm{M}(\kappa, F_1\kappa, t)} - 1 \right), \left(\frac{1}{\mathrm{M}\left(x_{2m+1}, x_{2m+2}, t\right)} - 1 \right), \\ \left(\frac{1}{\mathrm{M}\left(x_{2m+1}, F_1\kappa, t\right)} - 1 \right), \left(\frac{1}{\mathrm{M}\left(\kappa, x_{2m+2}, t\right)} - 1 \right), \\ \frac{1}{2} \left(\frac{1}{\mathrm{M}(\kappa, F_1\kappa, t)} - 1 + \frac{1}{\mathrm{M}\left(x_{2m+1}, F_2 x_{2m+1}, t\right)} - 1 \right), \\ \frac{1}{2} \left(\frac{1}{\mathrm{M}\left(x_{2m+1}, F_1\kappa, t\right)} - 1 + \frac{1}{\mathrm{M}\left(\kappa, F_2 x_{2m+1}, t\right)} - 1 \right) \\ \to \beta \max \left\{ \frac{1}{\mathrm{M}(\kappa, F_1\kappa, t)} - 1, \frac{1}{2} \left(\frac{1}{\mathrm{M}(\kappa, F_1\kappa, t)} - 1 \right) \right\}, \text{ as } j \to \infty. \end{array} \right\}$$

And

$$\frac{1}{N\left(x_{2m+2}, F_1\kappa, t\right)} = \frac{1}{\mathrm{N}\left(F_2 x_{2m+1}, F_1\kappa, t\right)} \geq \alpha \left(\frac{1}{\mathrm{N}\left(x_{2m+1}, \kappa, t\right)} \right)$$

$$+\beta \min \left\{ \begin{array}{c} \left(\frac{1}{N\left(x_{2m+1}, \kappa, t\right)} \right), \\ \left(\frac{1}{\mathrm{N}(\kappa, F_1\kappa, t)} \right), \left(\frac{1}{\mathrm{N}\left(x_{2m+1}, F_2 x_{2m+1}, t\right)} \right), \\ \left(\frac{1}{\mathrm{N}\left(x_{2m+1}, F_1\kappa, t\right)} \right), \left(\frac{1}{\mathrm{N}\left(\kappa, F_2 x_{2m+1}, t\right)} \right), \\ \frac{1}{2} \left(\frac{1}{\mathrm{N}(\kappa, F_1\kappa, t)} + \frac{1}{\mathrm{N}\left(x_{2m+1}, F_2 x_{2m+1}, t\right)} \right) \\ \frac{1}{2} \left(\frac{1}{\mathrm{N}\left(x_{2m+1}, F_1\kappa, t\right)} + \frac{1}{\mathrm{N}\left(\kappa, F_2 x_{2m+1}, t\right)} \right) \end{array} \right\}$$

$$= \alpha \left(\frac{1}{\mathrm{N}\left(x_{2m+1}, \kappa, t\right)} \right)$$

$$+\beta \min \left\{ \begin{array}{c} \left(\frac{1}{\mathrm{N}\left(x_{2m+1}, \kappa, t\right)} \right), \left(\frac{1}{\mathrm{N}(\kappa, F_1\kappa, t)} \right), \left(\frac{1}{\mathrm{N}\left(x_{2m+1}, x_{2m+2}, t\right)} \right), \\ \left(\frac{1}{\mathrm{N}\left(x_{2m+1}, F_1\kappa, t\right)} \right), \left(\frac{1}{\mathrm{N}\left(\kappa, x_{2m+2}, t\right)} \right), \\ \frac{1}{2} \left(\frac{1}{\mathrm{N}(\kappa, F_1\kappa, t)} + \frac{1}{\mathrm{N}\left(x_{2m+1}, F_2 x_{2m+1}, t\right)} \right), \\ \frac{1}{2} \left(\frac{1}{\mathrm{N}\left(x_{2m+1}, F_1\kappa, t\right)} + \frac{1}{\mathrm{N}\left(\kappa, F_2 x_{2m+1}, t\right)} \right) \end{array} \right\}$$

$$\to \beta \min \left\{ \frac{1}{\mathrm{N}(\kappa, F_1\kappa, t)} - 1, \frac{1}{2} \left(\frac{1}{\mathrm{N}(\kappa, F_1\kappa, t)} - 1 \right) \right\}, \text{ as } j \to \infty.$$

Then,

$$\lim_{x \to \infty} \sup \left(\frac{1}{M\left(x_{2m+2}, F_1 K, t\right)} - 1 \right) \leq \beta \left(\frac{1}{\mathrm{M}(K, F_1 K, t)} - 1 \right) \text{ for } t > 0.$$

And

$\lim_{x \to \infty} \inf \left(\frac{1}{N\left(x_{2m+2}, F_1 K, t\right)} \right) \geq \beta \left(\frac{1}{\mathrm{N}(K, F_1 K, t)} \right)$ for $t > 0$

The above (XXI) is together with (XX) and (XIX), we get

$$\frac{1}{M(K, F_1 K, t)} - 1 \leq \beta \left(\frac{1}{\mathrm{M}(K, F_1 K, t)} - 1 \right) \text{ for } t > 0.$$

and

$$\frac{1}{N\left(K,F_1K,t\right)}\geq\beta\left(\frac{1}{\mathrm{N}\left(K,F_1K,t\right)}\right)\ \text{for } t>0$$

Since $(1-\beta)\neq 0$, therefore we get that $M\left(K,F_1,t\right)=1$ and $N\left(K,F_1,t\right)=0$, this implies that $F_1K=K$. Similarly, we can show $F_2K=K$. Hence proved that K is a common fixed point of F_1 and F_2, that is, $F_1K=F_2K=K$.

Uniqueness: let $\kappa^*\in X$ be the other common fixed point of F_1 and F_2 such that $F_1\kappa^*=F_2\kappa^*=\kappa^*$, then by the view of (I) and (II), for $t>0$, we have

$$\begin{aligned}\frac{1}{\mathrm{M}\left(\kappa,\kappa^*,t\right)}-1&=\left(\frac{1}{\mathrm{M}\left(F_1\kappa,F_2\kappa^*,t\right)}-1\right)\leq\alpha\left(\frac{1}{M\left(\kappa,K^*,t\right)}-1\right)\\&=(\alpha+\beta)\left(\frac{1}{M\left(\kappa,\kappa^*,t\right)}-1\right)=(\alpha+\beta)\left(\frac{1}{\mathrm{M}\left(F_1K_1,F_2K^*,t\right)}-1\right)\\\leq(\alpha+\beta)^2&\left(\frac{1}{\mathrm{M}\left(\kappa,\kappa^*,t\right)}-1\right)\leq\cdots\leq(\alpha+\beta)^m\left(\frac{1}{\mathrm{M}\left(\kappa,\kappa^*,t\right)}-1\right)\to 0,\ \text{as } m\to\infty.\end{aligned}$$

and

$$\begin{aligned}\frac{1}{\mathrm{N}\left(\kappa,\kappa^*,t\right)}&=\left(\frac{1}{\mathrm{N}\left(F_1\kappa,F_2\kappa^*,t\right)}\right)\geq\alpha\left(\frac{1}{N\left(\kappa,K^*,t\right)}\right)\\&=(\alpha+\beta)\left(\frac{1}{N\left(\kappa,\kappa^*,t\right)}\right)=(\alpha+\beta)\left(\frac{1}{\left(F_1K_1,F_2K^*,t\right)}\right)\\\geq(\alpha+\beta)^2&\left(\frac{1}{\mathrm{N}\left(\kappa,\kappa^*,t\right)}\right)\geq\cdots\geq(\alpha+\beta)^m\left(\frac{1}{\mathrm{N}\left(\kappa,\kappa^*,t\right)}\right)\to 0,\ \text{as } m\to\infty.\end{aligned}$$

Theorem 4.1. Let (X, M, N, $*$,$\diamond$) be a complete IFM space in which M is triangular and N is co-triangular and a pair of self mappings $f_1,f_2:X\to X$ satisfies,

$$\begin{aligned}&\frac{1}{\mathrm{M}\left(f_1x,f_2y,t\right)}-1\leq\alpha\left(\frac{1}{\mathrm{M}(x,y,t)}-1\right)\\&+\beta\max\left\{\begin{array}{c}\left(\frac{1}{M(x,y,t)}-1\right),\left(\frac{1}{\mathrm{M}(x,F_1x,t)}-1\right),\\\left(\frac{1}{\mathrm{M}(y,f_2y,t)}-1\right),\left(\frac{1}{\mathrm{M}(y,f_1x,t)}-1\right),\left(\frac{1}{\mathrm{M}(x,f_2y,t)}-1\right),\frac{1}{2}\left(\frac{1}{M(x,f_1y,t)}-1+\frac{1}{\mathrm{M}(x,F_2x,t)}-1\right),\\\frac{1}{2}\left(\frac{1}{\mathrm{M}(y,f_1x,t)}-1+\frac{1}{\mathrm{M}(x,f_2y,t)}-1\right)\end{array}\right\}\end{aligned}$$

and

$$\begin{aligned}&\frac{1}{\mathrm{N}\left(f_1x,f_2y,t\right)}-1\geq\alpha\left(\frac{1}{\mathrm{N}(x,y,t)}-1\right)\\&+\beta\min\left\{\begin{array}{c}\left(\frac{1}{N(x,y,t)}-1\right),\left(\frac{1}{\mathrm{N}(x,F_1x,t)}-1\right),\\\left(\frac{1}{\mathrm{N}(y,f_2y,t)}-1\right),\left(\frac{1}{\mathrm{N}(y,f_1x,t)}-1\right),\left(\frac{1}{\mathrm{N}(x,f_2y,t)}-1\right),\frac{1}{2}\left(\frac{1}{N(x,f_1y,t)}-1+\frac{1}{\mathrm{N}(x,F_2x,t)}-1\right),\\\frac{1}{2}\left(\frac{1}{\mathrm{N}(y,f_1x,t)}-1+\frac{1}{\mathrm{N}(x,f_2y,t)}-1\right)\end{array}\right\}\end{aligned}$$

for all $x,y\in X,t>0,\alpha\in(0,1)$ and $\beta\geq 0$ with $(\alpha+2\beta)<1$. Then f_1 and f_2 have a unique common fixed point in X.

Proof. Fix $x_0\in X$ and define a sequence $\{x_m\}$ in X such that

$$x_{2m+1}=f_1x_{2m}\ \text{and}\ x_{2m+2}=f_2x_{2m+1}\ \text{for } m\geq 0.$$

Now, from (1), for $t > 0$, we have

$$
\begin{aligned}
\frac{1}{M\left(x_{2m+1}, x_{2m+2}, t\right)} - 1 &= \frac{1}{M\left(f_1 x_{2m}, f_2 x_{2m+1}, t\right)} - 1 \\
&\leq \alpha\left(\frac{1}{\mathrm{M}\left(x_{2m}, x_{2m+1}, t\right)} - 1\right) + \\
\beta \max\Bigg\{ \left(\frac{1}{M\left(x_{2m}, x_{2m+1}, t\right)} - 1\right), &\left(\frac{1}{\mathrm{M}\left(\mathrm{x}, f_1 x_{2m}, t\right)} - 1\right), \left(\frac{1}{\mathrm{M}\left(x_{2m+1}, f_2 x_{2m+1}, t\right)} - 1\right), \\
\left(\frac{1}{\mathrm{M}\left(x_{2m+1}, f_1 x_{2m}, t\right)} - 1\right), &\left(\frac{1}{\mathrm{M}\left(x_{2m}, f_2 x_{2m+1}, t\right)} - 1\right), \frac{1}{2}\left(\frac{1}{\mathrm{M}\left(x_{2m}, f_1 x_{2m}, t\right)} - 1\right. \\
&\left. + \frac{1}{\mathrm{M}\left(x_{2m+1}, f_2 x_{2m+1}, t\right)} - 1\right), \frac{1}{2}\left(\frac{1}{\mathrm{M}\left(x_{2m+1}, f_1 x_{2m}, t\right)} - 1\right. \\
&\left. + \frac{1}{\mathrm{M}\left(x_{2m}, f_2 x_{2m+1}, t\right)} - 1\right\},
\end{aligned}
$$

and

$$
\begin{gathered}
\frac{1}{N\left(x_{2m+1}, x_{2m+2}, t\right)} - 1 = \frac{1}{N\left(f_1 x_{2m}, f_2 x_{2m+1}, t\right)} - 1 \geq \alpha\left(\frac{1}{\mathrm{N}\left(x_{2m}, x_{2m+1}, t\right)} - 1\right) \\
\left\{\begin{array}{c}
\left(\frac{1}{N\left(x_{2m}, x_{2m+1}, t\right)} - 1\right), \left(\frac{1}{\mathrm{N}\left(\mathrm{x}, f_1 x_{2m}, t\right)} - 1\right), \left(\frac{1}{\mathrm{N}\left(x_{2m+1}, f_2 x_{2m+1}, t\right)} - 1\right), \\
\left(\frac{1}{\mathrm{N}\left(x_{2m+1}, f_1 x_{2m}, t\right)} - 1\right), \left(\frac{1}{\mathrm{N}\left(x_{2m}, f_2 x_{2m+1}, t\right)} - 1\right), \\
\frac{1}{2}\left(\frac{1}{\mathrm{N}\left(x_{2m}, f_1 x_{2m}, t\right)} - 1 + \frac{1}{\mathrm{N}\left(x_{2m+1}, f_2 x_{2m+1}, t\right)} - 1\right), \\
\frac{1}{2}\left(\frac{1}{\mathrm{N}\left(x_{2m+1}, f_1 x_{2m}, t\right)} - 1 + \frac{1}{\mathrm{N}\left(x_{2m}, f_2 x_{2m+1}, t\right)} - 1\right.
\end{array}\right\}
\end{gathered}
$$

after simplification, we obtain

$$
\begin{aligned}
\frac{1}{M\left(x_{2m+1}, x_{2m+2}, t\right)} - 1 &= \alpha\left(\frac{1}{\mathrm{M}\left(x_{2m}, x_{2m+1}, t\right)} - 1\right) \\
&+ \beta \max\left\{\begin{array}{c}
\left(\frac{1}{\mathrm{M}\left(x_{2m}, x_{2m+1}, t\right)} - 1\right), \\
\left(\frac{1}{\mathrm{M}\left(x_{2m+1}, x_{2m+2}, t\right)} - 1\right), \left(\frac{1}{\mathrm{M}\left(x_{2m}, x_{2m+2}, t\right)} - 1\right), \\
\frac{1}{2}\left(\frac{1}{\mathrm{M}\left(x_{2m}, x_{2m+1}, t\right)} - 1 + \frac{1}{\mathrm{M}\left(x_{2m+1}, x_{2m+2}, t\right)} - 1\right),
\end{array}\right\}
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{1}{N\left(x_{2m+1}, x_{2m+2}, t\right)} - 1 &= \alpha\left(\frac{1}{\mathrm{N}\left(x_{2m}, x_{2m+1}, t\right)} - 1\right) \\
&+ \beta \min\left\{\begin{array}{c}
\left(\frac{1}{\mathrm{N}\left(x_{2m}, x_{2m+1}, t\right)} - 1\right), \\
\left(\frac{1}{\mathrm{N}\left(x_{2m+1}, x_{2m+2}, t\right)} - 1\right), \left(\frac{1}{\mathrm{N}\left(x_{2m}, x_{2m+2}, t\right)} - 1\right), \\
\frac{1}{2}\left(\frac{1}{\mathrm{N}\left(x_{2m}, x_{2m+1}, t\right)} - 1 + \frac{1}{\mathrm{N}\left(x_{2m+1}, x_{2m+2}, t\right)} - 1\right),
\end{array}\right\}
\end{aligned}
$$

Then, we may have the following four cases;

(i) If $\frac{1}{\mathrm{M}\left(x_{2m},x_{2m+1},t\right)}-1$ is the maximum and $\frac{1}{\mathrm{N}\left(x_{2m},x_{2m+1},t\right)}$ is minimum in (V), then after simplification for $t>0$, we obtain

$$\frac{1}{M\left(x_{2m+1},x_{2m+2},t\right)}-1\leq\lambda_1\left(\frac{1}{\mathrm{M}\left(x_{2m},x_{2m+1},t\right)}-1\right),\text{ where }\lambda_1=\alpha+\beta<1.$$

and

$$\frac{1}{N\left(x_{2m+1},x_{2m+2},t\right)}\geq\lambda_1\left(\frac{1}{\mathrm{N}\left(x_{2m},x_{2m+1},t\right)}\right),\text{ where }\lambda_1=\alpha+\beta>1$$

(ii) If $\frac{1}{M\left(x_{2m+1},x_{2m+2},t\right)}-1$ is the maximum and $\frac{1}{N\left(x_{2m+1},x_{2m+2},t\right)}$ is the minimum in (V), then after simplification for $t>0$, we obtain

$$\frac{1}{M\left(x_{2m+1},x_{2m+2},t\right)}-1\leq\lambda_2\left(\frac{1}{\mathrm{M}\left(\mathrm{x},x_{2m+1},t\right)}-1\right),\text{ where }\lambda_2=\frac{\alpha}{1-\beta}<1.$$

and

$$\frac{1}{N\left(x_{2m+1},x_{2m+2},t\right)}\geq\lambda_2\left(\frac{1}{\mathrm{N}\left(\mathrm{x},x_{2m+1},t\right)}\right),\text{ where }\lambda_2=\frac{\alpha}{1-\beta}>1.$$

(iii) If $\frac{1}{M\left(\mathrm{x},x_{2m+2},t\right)}-1$ is the maximum and $\frac{1}{N\left(\mathrm{x},x_{2m+2},t\right)}$ is the minimum in (V), then after simplification for $t>0$, we obtain

$$\frac{1}{M\left(x_{2m+1},x_{2m+2},t\right)}-1\leq\lambda_3\left(\frac{1}{\mathrm{M}\left(x_{2m},x_{2m+1},t\right)}-1\right),\text{ where }\lambda_3=\frac{\alpha+\beta}{1-\beta}<1.$$

and

$$\frac{1}{N\left(x_{2m+1},x_{2m+2},t\right)}\geq\lambda_3\left(\frac{1}{\mathrm{N}\left(x_{2m},x_{2m+1},t\right)}\right),\text{ where }\lambda_3=\frac{\alpha+\beta}{1-\beta}>1.$$

(iv) If $\frac{1}{2}\left(\frac{1}{M\left(x_{2m},x_{2m+1},t\right)}-1+\frac{1}{\mathrm{M}\left(x_{2m+1},x_{2m+2},t\right)}-1\right)$ is the maximum and $\frac{1}{2}\left(\frac{1}{N\left(x_{2m},x_{2m+1},t\right)}-\ 1+\frac{1}{\mathrm{N}\left(x_{2m+1},x_{2m+2},t\right)}-1\right)$ is the minimum in (V), then after simplification for $t>0$, we obtain

$$\frac{1}{M\left(x_{2m+1},x_{2m+2},t\right)}-1\leq\lambda_4\left(\frac{1}{\mathrm{M}\left(x_{2m},x_{2m+1},t\right)}-1\right),\text{ where }\lambda_4=\frac{2\alpha+\beta}{2-\beta}<1.$$

and

$$\frac{1}{N\left(x_{2m+1},x_{2m+2},t\right)}-1\geq\lambda_4\left(\frac{1}{\mathrm{N}\left(x_{2m},x_{2m+1},t\right)}-1\right),\text{ where }\lambda_4=\frac{2\alpha+\beta}{2-\beta}>1.$$

Let us define $\mu_1:=\max\{\lambda_1,\lambda_2,\lambda_3,\lambda_4\}<1$, then from (VI)-(X), we get that

$$\frac{1}{M\left(x_{2m+1},x_{2m+2},t\right)}-1\leq\mu_1\left(\frac{1}{\mathrm{M}\left(x_{2m},x_{2m+1},t\right)}-1\right)\text{ for }t>0.$$

and

$$\frac{1}{N\left(x_{2m+1},x_{2m+2},t\right)}\geq\mu_1\left(\frac{1}{\mathrm{N}\left(x_{2m},x_{2m+1},t\right)}\right)\ \text{for } t<0.$$

Similarly, again by the view of (I and II), for $t>0$ and $t<0$, we have

$$\frac{1}{M\left(x_{2m+2},x_{2m+3},t\right)}-1=\frac{1}{M\left(F_1x_{2m+2},F_2x_{2m+1},t\right)}-1\leq\alpha\left(\frac{1}{\mathrm{M}\left(x_{2m+2},x_{2m+1},t\right)}-1\right)$$
$$+\beta\max\left\{\begin{array}{c}\left(\frac{1}{\mathrm{M}\left(x_{2m+2},x_{2m+1},t\right)}-1\right),\\ \left(\frac{1}{\mathrm{M}\left(x_{2m+2},F_1x_{2m+2},t\right)}-1\right),\left(\frac{1}{\mathrm{M}\left(x_{2m+1},F_2x_{2m+1},t\right)}-1\right),\\ \left(\frac{1}{\mathrm{M}\left(x_{2m+1},F_1x_{2m+2},t\right)}-1\right),\left(\frac{1}{\mathrm{M}\left(x_{2m+2},F_2x_{2m+1},t\right)}-1\right),\\ \frac{1}{2}\left(\frac{1}{\mathrm{M}\left(x_{2m+2},F_1x_{2m+2},t\right)}-1+\frac{1}{\mathrm{M}\left(x_{2m+1},F_2x_{2m+1},t\right)}-1\right),\\ \frac{1}{2}\left(\frac{1}{\mathrm{M}\left(x_{2m+1},F_1x_{2m+2},t\right)}-1+\frac{1}{\mathrm{M}\left(x_{2m+2},F_2x_{2m+1},t\right)}-1\right)\end{array}\right\},$$

and

$$\frac{1}{N\left(x_{2m+2},x_{2m+3},t\right)}=\frac{1}{N\left(F_1x_{2m+2},F_2x_{2m+1},t\right)}\geq\alpha\left(\frac{1}{\mathrm{N}\left(x_{2m+2},x_{2m+1},t\right)}\right)$$
$$+\beta\min\left\{\begin{array}{c}\left(\frac{1}{\mathrm{N}\left(x_{2m+2},x_{2m+1},t\right)}\right),\\ \left(\frac{1}{\mathrm{N}\left(x_{2m+2},F_1x_{2m+2},t\right)}\right),\left(\frac{1}{\mathrm{N}\left(x_{2m+1},F_2x_{2m+1},t\right)}\right),\\ \frac{1}{\mathrm{N}\left(x_{2m+1},F_1x_{2m+2},t\right)}\Bigg),\left(\frac{1}{\mathrm{N}\left(x_{2m+2},F_2x_{2m+1},t\right)}\right),\\ \frac{1}{2}\left(\frac{1}{\mathrm{N}\left(x_{2m+2},F_1x_{2m+2},t\right)}+\frac{1}{\mathrm{N}\left(x_{2m+1},F_2x_{2m+1},t\right)}\right)\\ \frac{1}{2}\left(\frac{1}{\mathrm{N}\left(x_{2m+1},F_1x_{2m+2},t\right)}+\frac{1}{\mathrm{N}\left(x_{2m+2},F_2x_{2m+1},t\right)}\right)\end{array}\right\},\ ,$$

after simplification, we get that

$$\frac{1}{M\left(x_{2m+2},x_{2m+3},t\right)}-1=\alpha\left(\frac{1}{\mathrm{M}\left(x_{2m+1},x_{2m+2},t\right)}-1\right)$$
$$+\beta\max\left\{\begin{array}{c}\left(\frac{1}{\mathrm{M}\left(x_{2m+1},x_{2m+2},t\right)}-1\right),\\ \left(\frac{1}{\mathrm{M}\left(x_{2m+2},x_{2m+3},t\right)}-1\right),\left(\frac{1}{\mathrm{M}\left(x_{2m+1},x_{2m+3},t\right)}-1\right),\\ \frac{1}{2}\left(\frac{1}{\mathrm{M}\left(x_{2m+2},x_{2m+3},t\right)}-1+\frac{1}{\mathrm{M}\left(x_{2m+1},x_{2m+2},t\right)}-1\right),\end{array}\right\}.$$

and

$$\frac{1}{N\left(x_{2m+2},x_{2m+3},t\right)}=\alpha\left(\frac{1}{\mathrm{N}\left(x_{2m+1},x_{2m+2},t\right)}\right)$$
$$+\beta\min\left\{\begin{array}{c}\left(\frac{1}{\mathrm{N}\left(x_{2m+1},x_{2m+2},t\right)}\right),\\ \left(\frac{1}{\mathrm{N}\left(x_{2m+2},x_{2m+3},t\right)}\right),\left(\frac{1}{\mathrm{N}\left(x_{2m+1},x_{2m+3},t\right)}\right),\\ \frac{1}{2}\left(\frac{1}{\mathrm{N}\left(x_{2m+2},x_{2m+3},t\right)}+\frac{1}{\mathrm{N}\left(x_{2m+1},x_{2m+2},t\right)}\right),\end{array}\right\}.$$

Again we may have the following four cases;

(i) If $\left(\frac{1}{M(x_{2m+1},x_{2m+2},t)}-1\right)$ is the maximum and $\left(\frac{1}{N(x_{2m+1},x_{2m+2},t)}\right)$ is the minimum term in (XII), then after simplification for $t>0$ and $t<0$, we obtain

$$\frac{1}{M\left(x_{2m+2},x_{2m+3},t\right)}-1\leq\lambda_1\left(\frac{1}{\mathrm{M}\left(x_{2m+1},x_{2m+2},t\right)}-1\right),\ \text{where } \lambda_1=\alpha+\beta<1.$$

and

$$\frac{1}{N\left(x_{2m+2},x_{2m+3},t\right)}\geq\lambda_1\left(\frac{1}{\mathrm{N}\left(x_{2m+1},x_{2m+2},t\right)}\right),\ \text{where } \lambda_1=\alpha+\beta>1$$

(ii) If $\left(\frac{1}{M(x_{2m+2},x_{2m+3},t)}-1\right)$ is the maximum and $\left(\frac{1}{N(x_{2m+2},x_{2m+3},t)}\right)$ is minimum term in (XII), then after simplification for $t>0$ and $t<0$, we obtain

$$\frac{1}{M\left(x_{2m+2},x_{2m+3},t\right)}-1\leq\lambda_2\left(\frac{1}{\mathrm{M}\left(x_{2m+1},x_{2m+2},t\right)}-1\right),\ \text{where } \lambda_2=\frac{\alpha}{1-\beta}<1$$

and

$$\frac{1}{N\left(x_{2m+2},x_{2m+3},t\right)}\geq\lambda_2\left(\frac{1}{\mathrm{N}\left(x_{2m+1},x_{2m+2},t\right)}\right),\ \text{where } \lambda_2=\frac{\alpha}{1-\beta}>1$$

(iii) If $\left(\frac{1}{M(x_{2m+1},x_{2m+3},t)}-1\right)$ is the maximum and $\left(\frac{1}{M(x_{2m+1},x_{2m+3},t)}\right)$ is the minimum term in (XII), then after simplification for $t>0$ and $t<0$, we obtain

$$\frac{1}{M\left(x_{2m+2},x_{2m+3},t\right)}-1\leq\lambda_3\left(\frac{1}{\mathrm{M}\left(x_{2m+1},x_{2m+2},t\right)}-1\right),\ \text{where } \lambda_3=\frac{\alpha+\beta}{1-\beta}<1.$$

and

$$\frac{1}{N\left(x_{2m+2},x_{2m+3},t\right)}\geq\lambda_3\left(\frac{1}{\mathrm{N}\left(x_{2m+1},x_{2m+2},t\right)}\right),\ \text{where } \lambda_3=\frac{\alpha+\beta}{1-\beta}>1.$$

(iv) If $\frac{1}{2}\left(\frac{1}{M(x_{2m+2},x_{2m+3},t)}-1+\frac{1}{\mathrm{M}(x_{2m+1},x_{2m+2},t)}-1\right)$ is the maximum and $\frac{1}{2}\left(\frac{1}{N(x_{2m+2},x_{2m+3},t)}+\frac{1}{\mathrm{N}(x_{2m+1},x_{2m+2},t)}\right)$ is the minimum term in (XII), then after simplification for $t>0$ and $t<0$, we obtain

$$\frac{1}{M\left(x_{2m+2},x_{2m+3},t\right)}-1\leq\lambda_4\left(\frac{1}{\mathrm{M}\left(x_{2m+1},x_{2m+2},t\right)}-1\right),\ \text{where } \lambda_4=\frac{2\alpha+\beta}{2-\beta}<1.$$

and

$$\frac{1}{N\left(x_{2m+2},x_{2m+3},t\right)}\geq\lambda_4\left(\frac{1}{\mathrm{N}\left(x_{2m+1},x_{2m+2},t\right)}\right),\ \text{where } \lambda_4=\frac{2\alpha+\beta}{2-\beta}>1.$$

Hence, from (XII)-(XIV), we get that

$$\frac{1}{M\left(x_{2m+2},x_{2m+3},t\right)}-1\leq\mu_1\left(\frac{1}{\mathrm{M}\left(x_{2m+1},x_{2m+2},t\right)}-1\right)\ \text{for } t>0$$

and

$$\frac{1}{N\left(x_{2m+2}, x_{2m+3}, t\right)} \geq \mu_1\left(\frac{1}{\mathrm{N}\left(x_{2m+1}, x_{2m+2}, t\right)}\right) \text{ for } t<0$$

where $\mu_1=\max\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}<1$.
Now from (XI) and (XVII), we have that

$$\begin{aligned}\frac{1}{M\left(x_{2m+2}, x_{2m+3}, t\right)}-1 &\leq \mu_1\left(\frac{1}{\mathrm{M}\left(x_{2m+1}, x_{2m+2}, t\right)}-1\right)\\ &\leq\left(\mu_1\right)^2\left(\frac{1}{\mathrm{M}\left(x_{2m}, x_{2m+1}, t\right)}-1\right) \quad \leq \cdots \leq\left(\mu_1\right)^{2m+2}\left(\frac{1}{\mathrm{M}\left(\mathrm{x}, x_1, t\right)}-1\right) \rightarrow 0,\\ &\text{as } m \rightarrow \infty.\end{aligned}$$

and

$$\begin{aligned}\frac{1}{N\left(x_{2m+2}, x_{2m+3}, t\right)} &\geq \mu_1\left(\frac{1}{\mathrm{N}\left(x_{2m+1}, x_{2m+2}, t\right)}\right)\\ &\geq\left(\mu_1\right)^2\left(\frac{1}{\mathrm{N}\left(x_{2m}, x_{2m+1}, t\right)}\right) \quad \geq \cdots \geq\left(\mu_1\right)^{2m+2}\left(\frac{1}{\mathrm{N}\left(\mathrm{x}, x_1, t\right)}\right) \rightarrow 0,\\ &\text{as } m \rightarrow \infty.\end{aligned}$$

Hence proved that $\{x_m\}_{m\geq 0}$ is a intuitionistic fuzzy contractive sequence, therefore

$$\lim_{m \rightarrow \infty} M\left(x_m, x_{m+1}, t\right)=1 \text{ for } t>0.$$

and

$$\lim_{m \rightarrow \infty} N\left(x_m, x_{m+1}, t\right)=1 \text{ for } t<0$$

Since M is triangular and N is co-triangular, for $k>m$ and $t>0, t<0$, then we have

$$\begin{aligned}&\frac{1}{M\left(x_m, x_k, t\right)}-1\\ &\leq\left(\frac{1}{\mathrm{M}\left(x_m, x_{m+1}, t\right)}-1\right)+\left(\frac{1}{\mathrm{M}\left(x_{m+1}, x_{m+2}, t\right)}-1\right)+\cdots+\left(\frac{1}{\mathrm{M}\left(x_{k-1}, x_k, t\right)}-1\right)\\ &\leq\left(\left(\mu_1\right)^m+\left(\mu_1\right)^{m+1}+\cdots+\left(\mu_1\right)^{k-1}\right)\left(\frac{1}{\mathrm{M}\left(x_0, x_1, t\right)}-1\right)\\ &\leq\left(\frac{\left(\mu_1\right)^m}{1-\mu_1}\right)\left(\frac{1}{M\left(x_0, x_1, t\right)}-1\right) \rightarrow 0, \text{ as } m \rightarrow \infty,\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{N\left(x_m, x_k, t\right)}\\
&\geq\left(\frac{1}{\mathrm{N}\left(x_m, x_{m+1}, t\right)}\right)+\left(\frac{1}{\mathrm{N}\left(x_{m+1}, x_{m+2}, t\right)}\right)+\cdots+\left(\frac{1}{\mathrm{N}\left(x_{k-1}, x_k, t\right)}\right)\\
&\geq\left(\left(\mu_1\right)^m+\left(\mu_1\right)^{m+1}+\cdots+\left(\mu_1\right)^{k-1}\right)\left(\frac{1}{\mathrm{N}\left(x_0, x_1, t\right)}\right)\\
&\geq\left(\frac{\left(\mu_1\right)^m}{1-\mu_1}\right)\left(\frac{1}{N\left(x_0, x_1, t\right)}\right) \rightarrow 1, \text{ as } m \rightarrow \infty,
\end{aligned}$$

which shows that $\{x_m\}$ is a cauchy sequence. By the completeness of $(X, M, \mathrm{N}, *, \diamond)$, $\exists \kappa \in X$ such that

$$\lim_{m \rightarrow \infty} M\left(\kappa, x_m, t\right)=1 \text{ for } t>0$$

and

$$\lim_{m \rightarrow \infty} N\left(\kappa, x_m, t\right)=0 \text{ for } t<0$$

Now we have to show that $F_1 \kappa=\kappa$. Since, M is triangular and N is Co - triangular, therefore

$$\frac{1}{M\left(\kappa, F_1 \kappa, t\right)}-1 \leq\left(\frac{1}{\mathrm{M}\left(\kappa, x_{2m+2}, t\right)}-1\right)+\left(\frac{1}{\mathrm{M}\left(x_{2m+2}, F_1 \kappa, t\right)}-1\right) \text{ for } t>0$$

and
$\frac{1}{N\left(\kappa, F_1 \kappa, t\right)} \geq\left(\frac{1}{\mathrm{N}\left(\kappa, x_{2m+2}, t\right)}\right)+\left(\frac{1}{\mathrm{N}\left(x_{2m+2}, F_1 \kappa, t\right)}\right)$ for $t<0 \ldots\ldots$ (XX)
Now, by the view of (I), (II), (XVIII) and (XIX) for $t>0$, we have that

$$\frac{1}{M\left(x_{2m+2}, F_1 \kappa, t\right)}-1=\frac{1}{\mathrm{M}\left(F_2 x_{2m+1}, F_1 \kappa, t\right)}-1 \leq \alpha\left(\frac{1}{\mathrm{M}\left(x_{2m+1}, \kappa, t\right)}-1\right)$$

$$+\beta \max \left\{\begin{array}{c}
\left(\frac{1}{M\left(x_{2m+1}, \kappa, t\right)}-1\right), \\
\left(\frac{1}{\mathrm{M}\left(\kappa, F_1 \kappa, t\right)}-1\right),\left(\frac{1}{\mathrm{M}\left(x_{2m+1}, F_2 x_{2m+1}, t\right)}-1\right), \\
\left(\frac{1}{\mathrm{M}\left(x_{2m+1}, F_1 \kappa, t\right)}-1\right),\left(\frac{1}{\mathrm{M}\left(\kappa, F_2 x_{2m+1}, t\right)}-1\right), \\
\frac{1}{2}\left(\frac{1}{\mathrm{M}\left(\kappa, F_1 \kappa, t\right)}-1+\frac{1}{\mathrm{M}\left(x_{2m+1}, F_2 x_{2m+1}, t\right)}-1\right), \\
\frac{1}{2}\left(\frac{1}{\mathrm{M}\left(x_{2m+1}, F_1 \kappa, t\right)}-1+\frac{1}{\mathrm{M}\left(\kappa, F_2 x_{2m+1}, t\right)}-1\right)
\end{array}\right\}$$

$$+\beta \max \left\{ \begin{array}{c} = \alpha \left(\frac{1}{\mathrm{M}(x_{2m+1},\kappa,t)} - 1 \right) \\ \left(\frac{1}{\mathrm{M}(x_{2m+1},\kappa,t)} - 1 \right), \left(\frac{1}{\mathrm{M}(\kappa,F_1\kappa,t)} - 1 \right), \left(\frac{1}{\mathrm{M}(x_{2m+1},x_{2m+2},t)} - 1 \right), \\ \left(\frac{1}{\mathrm{M}(x_{2m+1},F_1\kappa,t)} - 1 \right), \left(\frac{1}{\mathrm{M}(\kappa,x_{2m+2},t)} - 1 \right), \\ \frac{1}{2} \left(\frac{1}{\mathrm{M}(\kappa,F_1\kappa,t)} - 1 + \frac{1}{\mathrm{M}(x_{2m+1},F_2x_{2m+1},t)} - 1 \right), \\ \frac{1}{2} \left(\frac{1}{\mathrm{M}(x_{2m+1},F_1\kappa,t)} - 1 + \frac{1}{\mathrm{M}(\kappa,F_2x_{2m+1},t)} - 1 \right) \\ \to \beta \max \left\{ \frac{1}{\mathrm{M}(\kappa,F_1\kappa,t)} - 1, \frac{1}{2} \left(\frac{1}{\mathrm{M}(\kappa,F_1\kappa,t)} - 1 \right) \right\}, \text{ as } j \to \infty. \end{array} \right\}$$

and

$$\frac{1}{N(x_{2m+2},F_1\kappa,t)} = \frac{1}{\mathrm{N}(F_2x_{2m+1},F_1\kappa,t)} \geq \alpha \left(\frac{1}{\mathrm{N}(x_{2m+1},\kappa,t)} \right)$$

$$+\beta \min \left\{ \begin{array}{c} \left(\frac{1}{N(x_{2m+1},\kappa,t)} \right), \\ \left(\frac{1}{\mathrm{N}(\kappa,F_1\kappa,t)} \right), \left(\frac{1}{\mathrm{N}(x_{2m+1},F_2x_{2m+1},t)} \right), \\ \left(\frac{1}{\mathrm{N}(x_{2m+1},F_1\kappa,t)} \right), \left(\frac{1}{\mathrm{N}(\kappa,F_2x_{2m+1},t)} \right), \\ \frac{1}{2} \left(\frac{1}{\mathrm{N}(\kappa,F_1\kappa,t)} + \frac{1}{\mathrm{N}(x_{2m+1},F_2x_{2m+1},t)} \right) \\ \frac{1}{2} \left(\frac{1}{\mathrm{N}(x_{2m+1},F_1\kappa,t)} + \frac{1}{\mathrm{N}(\kappa,F_2x_{2m+1},t)} \right) \end{array} \right\}$$

$$= \alpha \left(\frac{1}{\mathrm{N}(x_{2m+1},\kappa,t)} \right)$$

$$+\beta \min \left\{ \begin{array}{c} \left(\frac{1}{\mathrm{N}(x_{2m+1},\kappa,t)} \right), \left(\frac{1}{\mathrm{N}(\kappa,F_1\kappa,t)} \right), \left(\frac{1}{\mathrm{N}(x_{2m+1},x_{2m+2},t)} \right), \\ \left(\frac{1}{\mathrm{N}(x_{2m+1},F_1\kappa,t)} \right), \left(\frac{1}{\mathrm{N}(\kappa,x_{2m+2},t)} \right), \\ \frac{1}{2} \left(\frac{1}{\mathrm{N}(\kappa,F_1\kappa,t)} + \frac{1}{\mathrm{N}(x_{2m+1},F_2x_{2m+1},t)} \right), \\ \frac{1}{2} \left(\frac{1}{\mathrm{N}(x_{2m+1},F_1\kappa,t)} + \frac{1}{\mathrm{N}(\kappa,F_2x_{2m+1},t)} \right) \end{array} \right\}$$

$$= \alpha \left(\frac{1}{\mathrm{N}(x_{2m+1},\kappa,t)} \right)$$

$$+\beta \min \left\{ \begin{array}{c} \left(\frac{1}{\mathrm{N}(x_{2m+1},\kappa,t)} \right), \left(\frac{1}{\mathrm{N}(\kappa,F_1\kappa,t)} \right), \left(\frac{1}{\mathrm{N}(x_{2m+1},x_{2m+2},t)} \right), \\ \left(\frac{1}{\mathrm{N}(x_{2m+1},F_1\kappa,t)} \right), \left(\frac{1}{\mathrm{N}(\kappa,x_{2m+2},t)} \right), \\ \frac{1}{2} \left(\frac{1}{\mathrm{N}(\kappa,F_1\kappa,t)} + \frac{1}{\mathrm{N}(x_{2m+1},F_2x_{2m+1},t)} \right), \\ \frac{1}{2} \left(\frac{1}{\mathrm{N}(x_{2m+1},F_1\kappa,t)} + \frac{1}{\mathrm{N}(\kappa,F_2x_{2m+1},t)} \right) \end{array} \right\}$$

$$\lim_{x\to\infty} \sup \left(\frac{1}{M(x_{2m+2},F_1K,t)} - 1 \right) \leq \beta \left(\frac{1}{\mathrm{M}(K,F_1K,t)} - 1 \right) \text{ for } t > 0$$

And

$\lim_{x\to\infty}\inf\left(\frac{1}{N\left(x_{2m+2},F_1K,t\right)}\right)\geq\beta\left(\frac{1}{\mathrm{N}(K,F_1K,t)}\right)$ for $t>0$.

The above (XXI) is together with (XX) and (XIX), we get

$$\frac{1}{M\left(K,F_1K,t\right)}-1\leq\beta\left(\frac{1}{\mathrm{M}\left(K,F_1K,t\right)}-1\right)\ \text{for } t>0.$$

$$\frac{1}{N\left(K,F_1K,t\right)}\geq\beta\left(\frac{1}{\mathrm{N}\left(K,F_1K,t\right)}\right)\ \text{for } t>0$$

Since $(1-\beta)\neq 0$, therefore we get that $M\left(K,F_1,t\right)=1$ and $N\left(K,F_1,t\right)=0$, this implies that $F_1K=K$. Similarly, we can show $F_2K=K$. Hence proved that K is a common fixed point of F_1 and F_2, that is, $F_1K=F_2K=K$.

Uniqueness: let $\kappa^*\in X$ be the other common fixed point of F_1 and F_2 such that $F_1\kappa^*=F_2\kappa^*=\kappa^*$, then by the view of (I) and (II), for $t>0$, we have

$$\begin{aligned}&\frac{1}{\mathrm{M}\left(\kappa,\kappa^*,t\right)}-1=\left(\frac{1}{\mathrm{M}\left(F_1\kappa,F_2\kappa^*,t\right)}-1\right)\leq\alpha\left(\frac{1}{M\left(\kappa,K^*,t\right)}-1\right)\\&=(\alpha+\beta)\left(\frac{1}{M\left(\kappa,\kappa^*,t\right)}-1\right)=(\alpha+\beta)\left(\frac{1}{\mathrm{M}\left(F_1K_1,F_2K^*,t\right)}-1\right)\\&\leq(\alpha+\beta)^2\left(\frac{1}{\mathrm{M}\left(\kappa,\kappa^*,t\right)}-1\right)\leq\cdots\leq(\alpha+\beta)^m\left(\frac{1}{\mathrm{M}\left(\kappa,\kappa^*,t\right)}-1\right)\to 0,\ \text{as } m\to\infty.\end{aligned}$$

and

$$\begin{aligned}&\frac{1}{\mathrm{N}\left(\kappa,\kappa^*,t\right)}=\left(\frac{1}{\mathrm{N}\left(F_1\kappa,F_2\kappa^*,t\right)}\right)\geq\alpha\left(\frac{1}{N\left(\kappa,K^*,t\right)}\right)\\&=(\alpha+\beta)\left(\frac{1}{N\left(\kappa,\kappa^*,t\right)}\right)=(\alpha+\beta)\left(\frac{1}{\left(F_1K_1,F_2K^*,t\right)}\right)\\&\geq(\alpha+\beta)^2\left(\frac{1}{\mathrm{N}\left(\kappa,\kappa^*,t\right)}\right)\geq\cdots\geq(\alpha+\beta)^m\left(\frac{1}{\mathrm{N}\left(\kappa,\kappa^*,t\right)}\right)\to 0,\ \text{as } m\to\infty.\end{aligned}$$

Hence we get that $M\left(K,K^*,t\right)=1$, and $N\left(K,K^*,t\right)=0$, this implies that $K=K^*$. Thus, F_1 and F_2 have a unique common fixed point in W.

4. **Conclusion**

This research demonstrates that generalized contraction theorems effectively solve fuzzy differential equations in intuitionistic fuzzy metric spaces, ensuring existence and uniqueness of solutions under flexible conditions. This approach advances the field by improving theoretical understanding and expanding practical applications in modeling uncertainty.

References

[1] L. A. Zadeh, Fuzzy sets, Information and Control, 8 (1965), 338353.
https://doi.org/10.1016/S0019-9958(65)90241-X

[2] I. Kramosil, J. Michalek, Fuzzy metric and statistical metric spaces, Kybernetika, 11 (1975), 336 344.

[3] A. George, P. Veeramani, On some results in fuzzy metric spaces, Fuzzy Set. Syst., 64 (1994), 395399. *https://doi.org/10.1016/0165-0114(94)90162-7.*

[4] M. Grabiec, Fixed points in fuzzy metric spaces, Fuzzy Set. Syst., 27 (1988), 385389.
https://doi.org/10.1016/0165-0114(88)90064-4.

[5] C. D. Bari, C. Vetro, A fixed point theorem for a family of mappings in fuzzy metric space, Rend. Circ. Mat. Palermo, 52 (2003), 315321.

[6] I. Beg, M. Abbas, Invariant approximation for fuzzy non expansive mappings, Math. Bohem., 136 (2011), 5159. https://doi.org/10.21136/MB.2011.141449

[7] I. Beg, S. Sedghi, N. Shobe, Fixed point theorems in fuzzy metric spaces, Int. J. Anal., 2013 (2013), 934145.
https://doi.org/10.1155/2013/934145

Ram Milan Singh
Department of Mathematics, Institute for Excellence in Higher Education Bhopal, India
Email: rammilansinghlig@gmail.com

Deepak Singh
Department of Mathematics, Swami Vivekanand University, Sagar-470001, Madhya Pradesh Bhopal, India
Email: deepaksinghresearch2023@gmail.com

Rahul Gourh
Research Scholar, Department of Mathematics Institute for Excellence in Higher Education Bhopal, India
Email:gourhrahi27@gmail.com