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Research Paper

RICCI-BOURGUIGNON SOLITON ON THREE DIMENSIONAL PARA-SASAKIAN MANIFOLD

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ABSTRACT

In the present paper we study Ricci-Bourguignon solitons on three dimensional para-Sasakian manifolds with potential vector field as a special vector field. We proved the conditions for such manifold to be isometric to hyperbolic space. Further, the nature of RB-soliton based on the value of real numbers ρ is investigated.

1. Introduction

Sato [14] introduced an almost paracontact manifold and Zamkovy [19] investigated the admittance of the pseudo-Riemannian metric with signature $(n+1, n)$ on the almost paracontact manifold of dimension $(2n+1)$. In general, an almost contact manifold has odd dimension, however, a paracontact manifold can also have even dimension. In modern differential geometry study of solitons was well explored. In recent years Ricci and Yamabe solitons were studied in Mathematics and Physics. De [5] recently investigated gradient solitons on a three-dimensional para-Sasakian manifold.

Bourguignon [1] initiated the study of Ricci-Bourguignon flow, later Catino et. al [2] developed Ricci-Bourguignon flow into parabolic theory in their research. Bourguignon introduced

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the Ricci -Bourguignon flow given by the equation

$$\frac{\partial g}{\partial t} = -2(S - \rho r g), g(0) = g_0, \tag{1.1}$$

where ρ is a non-zero real number, S is Ricci tensor and r is the scalar curvature. Ricci Bourguignon soliton (abbrivated as RB-soliton) is self similar solution of RB-flow and is defined on pseudo(semi)-Riemannian or Riemannaian manifold as follows:

$$(L_V g)(X, Y) = -2S(X, Y) + 2(\lambda + \rho r)g(X, Y), \tag{1.2}$$

where L_V is Lie-derivative along V, λ is soliton constant. If $\rho = 0$ then (1.2) defines Ricci soliton.

If the vector field V is gradient of potential function u then g is said to be gradient Ricci-Bourguignon soliton and the equation (1.2) takes the form

$$Hessu = -S(X, Y) + (\lambda + \rho r)g(X, Y), \tag{1.3}$$

where $Hessu$ denotes the Hessian of smooth function u on M and is defined by $Hessu = \nabla\nabla u$. Recently many authors studied RB-soliton on three-dimensional contact metric manifold ([3], [12], [4], [8], [16]). Pathra, Ali and Mofarreh[8] studied the RB-soliton and gradient RB-soliton on $k-$paracontact, (k, μ)-paracontact and para-Sasakian manifolds.

A vector field V on a pseudo-Riemannian manifold M is affine conformal, conformal, Jacobi type and torse-forming vector field if([11], [18], [6])

$$(L_V \nabla)(X, Y) = (Xf)Y + (Yf)X - g(X, Y)Df, \tag{1.4}$$

$$(L_V g)(X, Y) = 2\delta g(X, Y), \tag{1.5}$$

$$\nabla_X \nabla_Y V - \nabla_{\nabla_X Y} V + R(V, X)Y = 0, \tag{1.6}$$

$$\nabla_X V = fX + \alpha(X)V, \tag{1.7}$$

where X, Y and Z are smooth vector fields on M, ∇ is Levi-Civita connection on M, f and δ are smooth functions on M and α is a 1-form on M.

If $\alpha = 0$ in (1.7) then V is concircular, $\alpha = 0$ and $f = 1$ in (1.7) then V is a concurrent vetor field, and if $\alpha \neq 0$ and $f = 0$ in (1.7) then V is recurrent.

If a (0, 2)-tensor B satisfies the following

$$(\nabla_X B)(Y, Z) + (\nabla_Y B)(Z, X) + (\nabla_Z B)(X, Y) = k(X)g(Y, Z) + k(Y)g(Z, X) + k(Z)g(X, Y) \tag{1.8}$$

for any vector fields X, Y and Z on M, then it is known as the conformal quadratic Killing tensor, which is a generalization of conformal Killing vector field, where k is a 1-form on M.

2. Preliminaries

A (2n+1)-dimesional smooth manifold M admits an almost paracontact structure (ϕ, ξ, η), if there is a (1,1)-tensor field ϕ, a vector field ξ called Reeb vector field and η is a 1-form satisfying the following conditions

$$\phi^2 X = X - \eta(X)\xi, \quad \eta(\xi) = 1. \tag{2.1}$$

From the definition it follows that $\phi \circ \xi = 0, \ \ \eta \circ \phi = 0$.
Tensor field ϕ induces an almost paracomplex structure on horizontal bundle $D = ker\eta$ i.e., eigen distribution D_+ and D_- corresponding to eigen values $+1$ and -1 of ϕ, respectively, have equal dimension. An almost paracontact manifold is normal if Nijenhuis tensor field $N_\phi = [\phi, \phi] - 2d\eta \times \xi$ vanishes identically.
If an almost paracontact manifold admits pseudo-Riemannian metric g such that

$$g(\phi X, \phi Y) = -g(X, Y) + \eta(X)\eta(Y), \tag{2.2}$$

where X and Y in $X(M)$, then (ϕ, ξ, η, g) is called almost paraconatct metric structure, and a manifold with this metric structure is called paracontact metric manifold, we have

$$\Phi(X, Y) = g(X, \phi Y) = d\eta(X, Y), \tag{2.3}$$

for vector fields X and Y on M, Φ is fundamental 2-form.
In a paracontact metric manifold, we define $h = \frac{1}{2}L_\xi\phi$, where h is symmetric trace free operator and L_ξ is Lie derivative along direction ξ which satisfies $h\xi = 0, tr(h) = 0, tr(h\phi) = 0$ and h anti-commutes with ϕ. We have

$$\nabla_X\xi = -\phi X + \phi hX. \tag{2.4}$$

A paracontact manifold (M, ϕ, ξ, η, g) for which ξ is Killing is said to be $K-$ parcontact manifold. A normal paracontact metric manifold is called a para-Sasakian manifold. Every para-Sasakian manifold is K-paraconatct but converse is true for dimension 3. In a para-Sasakian manifold the following hold:

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X, \tag{2.5}$$

$$(\nabla_X\phi)(Y) = -g(X, Y)\xi + \eta(Y)X, \tag{2.6}$$

$$\nabla_X\xi = -\phi X, \tag{2.7}$$

$$R(\xi, X)Y = -g(X, Y)\xi + \eta(Y)X, \tag{2.8}$$

$$S(X, \xi) = -(n-1)\eta(X), \tag{2.9}$$

where X and Y are vector fields on M, R is Riemannian curvature tensor, S is Ricci tensor defined by $S(X, Y) = g(QX, Y)$ and Q is Ricci operator.
The Riemannian curvature tensor in a three dimesional pseudo-Riemannian manifold is given by
(2.10)

$$R(X, Y)Z = g(Y, Z)QX - g(X, Z)QY + g(QY, Z)X - g(QX, Z)Y - \frac{r}{2}(g(Y, Z)X - g(X, Z)Y).$$

Putting $Y = Z = \xi$ in (2.10), using (2.5) and (2.9), we get

$$S(X, Y) = \frac{1}{2}((r+2)g(X, Y) - (r+6)\eta(X)\eta(Y)). \tag{2.11}$$

Indeed we have the following results.

Lemma 2.1. [7] *In a three dimensional para-Sasakian manifold, we have* (ξr)*=0.*

Lemma 2.2. [17] *On an n-dimesional pseudo-Riemannian manifold endowed with conformal vector field V, we have*

$$(L_V S)(X,Y) = -(n-2)g(\nabla_X D\delta, Y) + (\Delta\delta)g(X,Y). \tag{2.12}$$

$$L_V r = -2\delta r + 2(n-1)\Delta\delta. \tag{2.13}$$

Lemma 2.3. [17] *From Yano formula, we have the following*

$$(L_V \nabla_Z g - \nabla_Z L_V g - \nabla_{[Z,V]} g)(X,Y) = -g((L_V \nabla(Z,X)),Y) - g((L_V \nabla(Z,Y)),X), \tag{2.14}$$

$$(\nabla_Z L_V g)(X,Y) = ((L_V \nabla(Z,X)),Y) + g((L_V \nabla(Z,Y)),X), \tag{2.15}$$

$$2g((L_V \nabla)(X,Y),Z) = (\nabla_X L_V g)(Y,Z) + (\nabla_Y L_V g)(Z,X) - (\nabla_Z L_V g)(X,Y), \tag{2.16}$$

$$(L_V R)(X,Y)Z = (\nabla_X L_V \nabla)(Y,Z) - (\nabla_Y L_V \nabla)(X,Z), \tag{2.17}$$

for all X, Y and Z on M.

3. Special vector field on three dimensional para-Sasakian manifold

In this section we study affine conformal, conformal and torse forming vector fields on para-Sasakian manifold.

Theorem 3.1. *If a pseudo-Riemannian metric of a three-dimensional para-Sasakian manifold is a RB-soliton whose potential vector field is affine conformal vector field then the Ricci tensor S is conformal quadratic Killing tensor.*

Proof. Taking covariant derivative along Z in (1.2), we get

$$(\nabla_Z L_V g)(X,Y) = -2(\nabla_Z S)(X,Y) + 2\rho(Zr)g(X,Y). \tag{3.1}$$

Cyclic change of X, Y, Z and in (3.1), and the use of equations (2.16) and (2.17), give

$$\begin{aligned} g((L_V \nabla)(X,Y),Z) = &- (\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Y) + (\nabla_Z S)(X,Y) + \rho(Xr)g(Y,Z) \\ &+ \rho(Yr)g(X,Z) - \rho(Zr)g(X,Y). \end{aligned} \tag{3.2}$$

Using (1.4) in (2.15), we have

$$(\nabla_X L_V g)(Y,Z) = 2(Xf)g(Y,Z). \tag{3.3}$$

Making use of (3.3) in (3.1), we infer that

$$(\nabla_X S)(Y,Z) = -(Xf)g(Y,Z) + \rho(Xr)g(Y,Z). \tag{3.4}$$

Cyclically inter changing X, Y and Z in (3.4), and adding the equations, we achive that

$$\begin{aligned} &(\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) + (\nabla_Z S)(X,Y) \\ &= (X(\rho Dr - Df))g(Y,Z) + (Y(\rho Dr - Df))g(Z,X) + (Z(\rho Dr - Df))g(X,Y). \end{aligned} \tag{3.5}$$

If we set dual 1-form $\rho Dr - Df$ as k, then last equation becomes

$$\begin{aligned} &(\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) + (\nabla_Z S)(X,Y) \\ &= k(X)g(Y,Z) + k(Y)g(X,Z) + k(Z)g(X,Y), \end{aligned} \tag{3.6}$$

which shows S is conformal quadratic Killing tensor. This completes the proof. □

Theorem 3.2. *In a three dimesional para-Sasakian manifold, with pseudo-Riemannnian metric as RB-soliton if the potential vector field V is an affine conformal vector field then it is Jacobi along geodesic of ξ.*

Proof. Taking covariant derivative along Z in equation (2.11), we get

(3.7)
$$(\nabla_Z S)(X,Y) = \frac{(Zr)}{2}g(X,Y) - \frac{(Zr)}{2}\eta(X)\eta(Y) - \left(\frac{r}{2}+3\right)\{(\nabla_Z)(X)\eta(Y) + (\nabla_Z\eta)(Y)\eta(X)\}.$$

Making use of (2.11) and (3.7) in (3.2), we obtain

(3.8)
$$g((L_V\nabla)(X,Y),Z) = \left(\rho - \frac{1}{2}\right)\{(Xr)g(Y,Z) + (Yr)g(X,Z) - (Zr)g(X,Y)\} + \frac{1}{2}\{(Xr)\eta(Y)\xi + (Yr)\eta(X)\xi - \eta(X)\eta(Y)Dr\}.$$

On comparing (1.4) and (3.8), for all vector fields Z, we arrive at

(3.9)
$$(Xf)Y - (Yf)X - g(X,Y)Df = \left(\rho - \frac{1}{2}\right)\{(Xr)Y + (Yr)X - g(X,Y)Dr\} + \frac{1}{2}\{(Xr)\eta(Y)\xi + (Yr)\eta(X)\xi - \eta(X)\eta(Y)Dr\} - 2(\frac{r}{2}+3)\{\eta(Y)\phi X + \eta(X)\phi Y\}.$$

Setting $X = \xi$ in last equation, we get

(3.10)
$$(\xi f)Y - (Yf)\xi - \eta(Y)Df = \left(\rho - \frac{1}{2}\right)\{(\rho r)Y + (Yr)\xi - \eta(Y)Dr\} + \frac{1}{2}\{(\xi r)\eta(Y)\xi + (Yr)\xi - \eta(Y)Dr\} - 2\left(\frac{r}{2}+3\right)\phi Y.$$

Putting $Y = \xi$ in (3.10), we obtain

$$Df = -\rho Dr. \tag{3.11}$$

Using foregoing equation in (3.10), we get

$$\left(-2\rho + \frac{1}{2}\right)(Xr)Y - \frac{1}{2}((Xr)\eta(Y)\xi + g(X,Y)dr + \eta(X)\eta(Y)Dr - (Yr)\eta(X)\xi) + (r+6)\{\eta(Y)\phi X + \eta(X)\phi Y\} = 0. \tag{3.12}$$

Setting $X = Y = \xi$ in (3.12), we achive

$$Dr = 0. \tag{3.13}$$

On integrating (3.13), we get r=constant. Using this in (3.8), we arrive at

$$(L_V\nabla)(X,Y) = -(r+6)\{\eta(X)\phi Y + \eta(Y)\phi X\}. \tag{3.14}$$

Substituting ξ for X and Y in last equation, we get

$$(L_V\nabla)(\xi,\xi) = 0. \tag{3.15}$$

Taking $X = Y = \xi$ in (2.7), we get $\nabla_\xi \xi = 0$. Using last equation in (1.6), we infer that

$$\nabla_\xi \nabla_\xi V - \nabla_{\nabla_\xi \xi} V + R(V, \xi)\xi = 0. \tag{3.16}$$

Make use of (3.15) and (3.16) in (1.6), we obtain

$$\nabla_\xi \nabla_\xi V + R(V, \xi)\xi = 0. \tag{3.17}$$

i.e., the vector field V is Jacobi vector field along the geodesic of ξ. □

Remark 3.3. In a three dimesional para-Sasakian manifold admitting RB-soliton, the Reeb vector field is geodesic.

Proof. Letting $V = \xi$ in (1.2) and using (2.11), we get

$$g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi) = (2\lambda + 2\rho r - r - 2)g(X, Y) + (r + 6)\eta(X)\eta(Y). \tag{3.18}$$

Substituting ξ for X in (3.18), we obtain

$$g(\nabla_\xi \xi, Y) + g(\nabla_Y \xi, \xi) = (2\lambda + 2\rho r + 4)\eta(Y). \tag{3.19}$$

Setting $Y = \xi$ in (3.19), we have

$$\lambda = -(\rho r + 2). \tag{3.20}$$

Using (3.20) in (3.19), we achive

$$\nabla_\xi \xi = 0. \tag{3.21}$$

i.e., ξ is geodesic. □

Theorem 3.4. *If a three dimesional para-Sasakian manifold admits an RB-soliton whose potential vector field V is conformal and scalar curvature is harmonic then manifold is Einstein and is locally isometric to Hyperbolic space $H^3(-1)$.*

Proof. Taking Lie derivative of $g(\xi, \xi) = 1$, we get

$$(L_V g)(\xi, \xi) + 2\eta(L_V \xi) = 0. \tag{3.22}$$

Putting $X = Y = \xi$ in (1.2), we obtain

$$(L_V g)(\xi, \xi) = 4 + 2\lambda + 2\rho r. \tag{3.23}$$

Using (3.23) in (3.22), we arrive at

$$\eta(L_V \xi) = -(2 + \lambda + \rho r). \tag{3.24}$$

Take Lie derivative along V of $\eta(\xi) = 1$, we achive

$$(L_V \eta)(\xi) = -\eta(L_V \xi). \tag{3.25}$$

Making use of (3.25) in (3.24), we get

$$(L_V \eta)(\xi) = 2 + \lambda + \rho r. \tag{3.26}$$

Setting $X = Y = \xi$ in (1.5), we infer that

$$(L_V g)(\xi, \xi) = 2\delta. \tag{3.27}$$

On comparing (3.23) and (3.27), we obatin

$$\delta = 2 + \lambda + \rho r. \tag{3.28}$$

Make use of (3.28) in (2.12), we get

$$(L_V S)(X, Y) = -\rho g(\nabla_X Dr, Y) + \rho(\Delta r)g(X, Y) \tag{3.29}$$

and

$$L_V r = -4r - 2\lambda r - 2\rho r^2 + 4\rho\Delta r. \tag{3.30}$$

We take Lie derivative along V in (2.11) to obtain

$$\begin{aligned}(L_V S)(X, Y) = \frac{(L_V r)}{2}\{g(X, Y) - \eta(X)\eta(Y)\} + \left(\frac{r}{2} + 1\right)(L_V g)(X, Y)\\ - \left(\frac{r}{2} + 3\right)\{(L_V \eta)(X)\eta(Y) + (L_V \eta)(Y)\eta(X)\}.\end{aligned} \tag{3.31}$$

Using (3.29) and (3.30) in (3.31), we have

(3.32)

$$\begin{aligned}-\rho g(\nabla_X Dr, Y) + \rho(\Delta r)g(X, Y) =& (-2r + 2\rho\Delta r + 2\lambda + 2\rho r)g(X, Y) + (2r + \lambda r + \rho r^2\\ &- 2\rho\Delta r)\eta(X)\eta(Y) - (r + 2)\{\left(\frac{r}{2} + 1\right)g(X, Y)\\ &- \left(\frac{r}{2} + 3\right)\eta(X)\eta(Y)\}.\end{aligned}$$

Substituting $X = Y = \xi$ in foregoing equation and using Lemma 2.1 and equation (3.26), we have

$$\rho\Delta r = -(\lambda + \rho r + 3). \tag{3.33}$$

If the scalar curvature is harmonic, that is $\Delta r = 0$, then last equation becomes

$$\lambda = -(\rho r + 3). \tag{3.34}$$

Taking Lie derivative of $\eta(X) = g(X, \xi)$ along V, we get

$$(L_V \eta)(X) = (L_V g)(X, \xi) + g(X, L_V \xi). \tag{3.35}$$

Using (1.2) in (3.35), we achive

$$(L_V \eta)(X) = 4(2 + \lambda + \rho r)\eta(X) + g(X, L_V \xi). \tag{3.36}$$

Making use of equation (1.7) in (3.35), we obtain

$$(L_V \eta)(X) = 2\delta\eta(X) + g(X, L_V \xi). \tag{3.37}$$

On comapring (3.36) and (3.37), we arrive at

$$\delta\eta(X) = (2 + \lambda + \rho r)\eta(X). \tag{3.38}$$

Putting $X = \xi$ in last equation, we get

$$\delta = (2 + \lambda + \rho r). \tag{3.39}$$

Utilizing (3.34) in (3.39), we infer that

$$\delta = -1. \tag{3.40}$$

Using (1.7), (3.34) and (3.40) in (1.2), we get

$$S(X,Y) = -2g(X,Y). \tag{3.41}$$

i.e., Manifold M is Einstein and its scalar cuvature $r = -6$.
Utilizing (3.41) in (2.10), we achive

$$R(X,Y)Z = -\{g(X,Z)Y - g(Y,Z)X\}. \tag{3.42}$$

Thus M has constant scetional curvature -1.
Using $r = -6$ in (3.34), we get $\lambda = 6\rho - 3$. Therefore the soliton is shrinking, steady or expanding according as $\rho < \frac{1}{2}$, $\rho = \frac{1}{2}$ and $\rho > \frac{1}{2}$ respectively. □

Corollary 3.5. *In a three dimensional para-Sasakian manifold with the pseudo-Riemannian metric as RB-soliton, the potential vector field V as conformal vector field reduces to homothetic vector field and the manifold has quadratic Killing tensor.*

Proof. Utilizing equation (1.7) in (1.2), we arrive at

$$S(X,Y) = (\lambda + \rho r - \delta)g(X,Y). \tag{3.43}$$

Taking covariant derivative along Z in (3.43), we obtain

$$(\nabla_Z S)(X,Y) = (\rho(Zr) - (Z\delta))g(X,Y). \tag{3.44}$$

Cyclically changing along X , Y and Z in (3.44) and adding all obtained equations, we get

(3.45)
$$\begin{aligned}(\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) + (\nabla_Z S)(X,Y) =&(\rho(Xr) - (X\delta))g(Y,Z) + (\rho(Yr)\\ &- (Y\delta))g(X,Z) + (\rho(Zr) - (Z\delta))g(X,Y).\end{aligned}$$

Contracting (3.45) with respect to Y and Z, we obatin

$$2(Xr) = 5\rho(Xr) - 5(X\delta). \tag{3.46}$$

Replacing Z and Y by ξ in (3.45), we infer that

$$(\nabla_X S)(\xi,\xi) + 2(\nabla_\xi S)(X,\xi) = \rho(Xr) - 2(\xi\delta)\eta(X). \tag{3.47}$$

Setting $X = \xi$ in (3.47), we get

$$(\xi\delta) = 0. \tag{3.48}$$

Using (3.48) in (3.47), we achive that

$$(X\delta) = 0, \tag{3.49}$$

for all vector fields X, which shows that δ is constant.
Using (3.49) in (3.47), we arrive at

$$(5\rho - 2)(Xr) = 0. \tag{3.50}$$

We choose $(5\rho - 2) \neq 0$. Therefore $(Xr) = 0$, for all vector fields X. i.e., r is a constant. Putting these in (3.45), we get

$$(\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) + (\nabla_Z S)(X,Y) = 0, \tag{3.51}$$

which shows that S is a quadratic Killing tensor. □

Theorem 3.6. *If three dimesional para-Sasakian manifold with torse forming vector field ξ admits RB-soliton (g, V, λ) then scalar curvature $r = -6$ and soliton constant $\lambda = 6\rho - 3$. Further the soliton is shrinking, steady and expanding according as $\rho < \frac{1}{2}$, $\rho = \frac{1}{2}$ and $\rho > \frac{1}{2}$.*

Proof. Taking inner product of (2.1) with ξ, we get

$$\eta(\nabla_X \xi) = f\eta(X) + \alpha(X). \tag{3.52}$$

Utilizing (2.7) in (3.52), we obtain

$$\alpha(X) = -f\eta(X). \tag{3.53}$$

Putting $X = \xi$ in last equation, we infer that

$$\alpha(\xi) = -f. \tag{3.54}$$

Making use of (3.53) in (2.1), we arrive at

$$\nabla_X \xi = f(X - \eta(X)\xi). \tag{3.55}$$

Taking $V = \xi$ in (1.2), we have

$$g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi) = -2S(X, Y) + 2(\lambda + \rho r)g(X, Y). \tag{3.56}$$

Using (3.55) in (3.56), we achive

$$S(X, Y) = (\lambda + \rho r - f)g(X, Y) + f\eta(X)\eta(Y). \tag{3.57}$$

Taking covariant derivative along Z in (3.57), we arrive at

$$\begin{aligned}(\nabla_Z S)(X, Y) =& (\rho(Zr) - (Zf))g(X, Y) + (Zf)\eta(X)\eta(Y) + f^2\{g(X, Z)\eta(Y) + g(Y, Z)\eta(X) \\ &- 2f\eta(X)\eta(Y)\eta(Z)\}.\end{aligned} \tag{3.58}$$

Making cyclic change of X, Y, Z in (3.59) and addition of the three equations yield

$$\begin{aligned}(\nabla_X S)(Y, Z) + (\nabla_Y S)(Z, X) + (\nabla_Z S)(X, Y) =& (\rho(Xr) - (Xf))g(Y, Z) + (\rho(Yr) - (Yf))g(X, Z) \\ &+ (\rho(Zr) - (Zf))g(X, Y) + (Xf)\eta(Y)\eta(Z) \\ &+ (Yf)\eta(X)\eta(Z) + (Zf)\eta(X)\eta(Y) \\ &+ 2f^2\{g(X, Y)\eta(Z) + g(X, Z)\eta(Y) + g(Y, Z)\eta(X)\} \\ &- 6f^3\eta(X)\eta(Y)\eta(Z).\end{aligned} \tag{3.59}$$

Contracting (3.59) with respect to $Y = Z = e_i$, i=1 to 3, we arrive at

$$2(Xr) = 5\rho(Xr) - 4(Xf) + 2(\xi f)\eta(X) + 6f^2\eta(X) - 6f^3\eta(X). \tag{3.60}$$

Taking $X = \xi$ in (3.60) and using Lemma 2.1, we get

$$(\xi f) = 3f^2 - 3f^3. \tag{3.61}$$

Utilizing (3.61) in (3.60), we obtain

$$(2 - 5\rho)(Xr) = -4(Xf) + (12f^2 - 12f^3)\eta(X). \tag{3.62}$$

Putting $Y = Z = \xi$ in (3.59), we have

$$g((\nabla_X Q)(\xi), \xi) + 2g((\nabla_\xi Q)(\xi), X) = \rho(Xr) + 6f^2\eta(X) - 6f^3\eta(X). \tag{3.63}$$

Using (3.58) in (3.63), we obtain

$$\rho(\xi f)\eta(X) = (2f^2 - 2f^3)\eta(X). \tag{3.64}$$

Putting $X = \xi$ in (3.64), we infer that

$$\rho(\xi f) = 2f^2 - 2f^3. \tag{3.65}$$

Making use of (3.61) in (3.64), we achive

$$(3\rho - 2)(f^2 - f^3) = 0. \tag{3.66}$$

If $(3\rho - 2) \neq 0$ then $f^2 = f^3$ or $f = 1$. this in (3.55) reduces the vector field ξ to a torse forming vector field.
Setting $f = 1$ in (3.62), we arrive at

$$(Xr) = 0, \tag{3.67}$$

for all vector fields X. Therefore r is a constant.
Using (2.11) in (1.2), we get

$$(L_V g)(X, Y) = (2\rho + 2\lambda - r - 2)g(X, Y) + (r + 6)\eta(X)\eta(Y). \tag{3.68}$$

Utilizing (3.68) in (3.31), we infer that

$$\begin{aligned}(L_V S)(X, Y) = &\left(\frac{r}{2} + 1\right)(2\rho + 2\lambda - r - 2)g(X, Y) + \left(\frac{r}{2} + 1\right)(r + 6)\eta(X)\eta(Y)\\ &- \left(\frac{r}{2} + 1\right)\{(L_V\eta)(X)\eta(Y) + (L_V\eta(X))\eta(Y)\}.\end{aligned} \tag{3.69}$$

Taking covariant derivative along Z in (2.11) with ξ as torse formnig vector field and making use of (3.55) and $(Xr) = 0$, we achive

$$(\nabla_Z S)(X, Y) = -\left(\frac{r}{2} + 1\right)\{g(X, Z)\eta(Y) + g(Y, Z)\eta(X) - 2\eta(X)\eta(Y)\eta(Z)\}. \tag{3.70}$$

Taking covariant derivative of (1.2) with respect to Z, we get

$$(\nabla_Z L_V g)(X, Y) = -2(\nabla_Z S)(X, Y). \tag{3.71}$$

Cyclic inter change of X, Y, Z in (3.71) and use of (2.17) , gives

$$g((L_V\nabla)(X, Y), Z) = -(\nabla_X S)(Y, Z) - (\nabla_Y S)(Z,) + (\nabla_Z S)(X, Y). \tag{3.72}$$

Utilizing (3.70) in (3.72), we have

$$(L_V\nabla)(X, Y) = (r + 2)\{g(X, Y)\xi - \eta(X)\eta(Y)\xi\}. \tag{3.73}$$

Taking covariant derivative along Z in (3.73), we get

$$\begin{aligned}(\nabla_Z L_V\nabla)(X, Y) =&(r + 2)\{g(X, Y)Z - g(X, Y)\eta(Z)\xi - g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi\\ &- \eta(X)\eta(Y)Z + 3\eta(X)\eta(Y)\eta(Z)\xi\}.\end{aligned} \tag{3.74}$$

Using (3.74) in (2.17), we arrive at

$$(L_V R)(X, Y)Z = (r + 2)\{g(Y, Z)X - g(X, Z)Y + \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X\}. \tag{3.75}$$

Contracting (3.75) with respect to X, we get

$$(L_V S)(Y, Z) = 2(r + 2)\{g(Y, Z) - \eta(Y)\eta(Z)\}. \tag{3.76}$$

Setting $Y = Z = \xi$ in last equation, we obtain

$$(L_V S)(\xi, \xi) = 0. \tag{3.77}$$

Thus taking $X = Y = \xi$ in (3.69) and comparing the equation with (3.77), we arrive at

$$(r+6)(L_V \eta)(\xi) = \left(\frac{r}{2}+1\right)(2\rho r + 2\lambda - r - 2) + (\frac{r}{2}+2)(r+6). \tag{3.78}$$

Substituting $Y = \xi$ in (3.70), we get

$$(L_V g)(X, \xi) = (2\rho r + 2\lambda + 4)\eta(X). \tag{3.79}$$

Lie differentiation of $\eta(X) = g(X, \xi)$ with respect to V and using (3.79), yields

$$(L_V \eta)(X) = (2\rho r + 2\lambda + 4)\eta(X) + g(X, L_V \xi). \tag{3.80}$$

Again taking Lie derivative of $g(\xi, \xi) = 1$ with respect to V, we achive

$$(L_V g)(\xi, \xi) + 2\eta(L_V \xi) = 0. \tag{3.81}$$

Making use of (3.79) in (3.81), we have

$$\eta(L_V \xi) = -(\rho r + \lambda + 2). \tag{3.82}$$

Now taking $X = \xi$ in (3.80) , we arrive at

$$(L_V \eta)(\xi) = \rho r + \lambda + 2. \tag{3.83}$$

Using (3.83) in (3.78), we infer that

$$\lambda = \frac{1}{4}(r - 2 - 4\rho r). \tag{3.84}$$

Putting $V = \xi$ in (3.68) and using (2.7) by setting $X = Y = \xi$, we obtain

$$\lambda = -(\rho r + 2). \tag{3.85}$$

On comparing (3.84) and (3.85), we obatin

$$r = -6. \tag{3.86}$$

Using (3.86) in (3.84), we achive

$$\lambda = 6\rho - 2. \tag{3.87}$$

This completes the proof. □

Theorem 3.7. *The potential vector field of RB-soliton on three dimesional para-Sasakian manifold with ξ as torse forming vector field is Jacobi along geodesic of ξ.*

Proof. Putting $X = Y = \xi$ in (3.73), we get

$$(L_V \nabla)(\xi, \xi) = 0. \tag{3.88}$$

Using (3.88) in (1.6), we achive

$$\nabla_\xi \nabla_\xi V + R(V, \xi)\xi = 0. \tag{3.89}$$

i.e., the vector V is a Jacobi vector field along geodesic of ξ. □

4. Gradient RB-soliton on three dimensional para-Sasakian manifold

Theorem 4.1. *Let pseudo-Riemannian metric of three dimesional para-Sasakian manifold be a gradient RB-soliton. Then the manifold is locally isometric to the hyperbolic space* $H^3(-1)$.

Proof. From equation (1.3), we have

$$\nabla_X Du = (\lambda + \rho r)X - QX. \tag{4.1}$$

Using equation (2.11) in(4.1), we get

$$\nabla_X Du = (\lambda + \rho r - \frac{r}{2} - 1)X + \left(\frac{r}{2} + 3\right)\eta(X)\xi. \tag{4.2}$$

Taking covariant derivative along Y in (4.2), we obtain

$$\begin{aligned}\nabla_Y \nabla_X Du = (\rho(Yr) - \frac{1}{2}(Yr))X + (\lambda + \rho r - \frac{r}{2} - 1)\nabla_Y X \\ + \frac{(Yr)}{2}\eta(X)\xi + \left(\frac{r}{2} + 3\right)\{g(\nabla_Y \xi, X)\xi + \eta(\nabla_Y X)\xi + \eta(X)\nabla_Y \xi\}.\end{aligned} \tag{4.3}$$

Using (2.7) in (4.3), we get

$$\begin{aligned}\nabla_Y \nabla_X Du =&(\rho(Yr) - \frac{1}{2}(Yr))X + (\lambda + \rho r - \frac{r}{2} - 1)\nabla_Y X \\ &+ \frac{(Yr)}{2}\eta(X)\xi + \left(\frac{r}{2} + 3\right)\{-g(\phi Y, X)\xi + \eta(\nabla_Y X)\xi - \eta(X)\phi(Y)\xi\}.\end{aligned} \tag{4.4}$$

Repeated use of (4.2) and (4.4) in $R(X,Y)Du = \nabla_X \nabla_Y Du - \nabla_Y \nabla_X Du - \nabla_{[X,Y]} Du$, we obtain

(4.5)

$$\begin{aligned}R(X,Y)Du =&(\rho(Xr) - \frac{1}{2}(Xr))Y + \frac{(Xr)}{2}\eta(Y)\xi - (\rho(Yr) - \frac{1}{2}(Yr))g(X,Z) + \frac{(Xr)}{2}\eta(Y)\eta(Z) \\ &- \frac{(Yr)}{2}\eta(X)\eta(Z) + \left(\frac{r}{2} + 3\right)\{-2g(\phi X, Y)\eta(Z) - g(\phi X, Z)\eta(Y) + g(\phi Y, Z)\eta(X)\}.\end{aligned}$$

Taking $X = Z = \xi$ in (4.5), we achive

$$g(R(\xi, Y)Du, \xi) = (\rho(\xi r))\eta(Y) - \rho(Yr) + (\xi r)\eta(Y). \tag{4.6}$$

Making use of Lemma 2.1 in last equation, we get

$$g(R(\xi, Y)Du, \xi) = -\rho(Yr). \tag{4.7}$$

Contracting (4.5) with respect to X, we get

$$S(Y, Du) = \left(-2\rho + \frac{1}{2}\right)(Yr). \tag{4.8}$$

Replace X by Du in (2.11), we obtain

$$S(Y, Du) = \left(\frac{r}{2} + 1\right)g(Y, Du) - \left(\frac{r}{2} + 3\right)\eta(Y)(\xi u). \tag{4.9}$$

On comparing (4.8) and (4.9), we achive

$$\left(-2\rho + \frac{1}{2}\right)(Yr) = \left(\frac{r}{2} + 1\right)(Yu) - \left(\frac{r}{2} + 3\right)\eta(Y)(\xi u). \tag{4.10}$$

Taking $Y = \xi$ in last equation, we arrive at

$$(\xi u) = 0. \tag{4.11}$$

Take inner product with Du in (2.5), we infer that

$$g(R(X,Y)Du,\xi) = \eta(Y)(Xu) - \eta(X)(Yu). \tag{4.12}$$

Thus taking $X = \xi$ in (4.12), we have

$$g(R(\xi,Y)Du,\xi) = \eta(Y)(\xi u) - (Yu). \tag{4.13}$$

Replace Z by ξ in (4.5) and on comparing the equation with (4.13), we arrive

$$\rho(Xr)\eta(Y) - \rho(Yr)\eta(X) - 2\left(\frac{r}{2}+3\right)g(\phi X,Y) = \eta(Y)(Xu) - \eta(X)(Yu). \tag{4.14}$$

Replace X by ϕX and Y by ϕY in (4.14) and using (2.1), we get

$$(r+6)g(X,\phi Y) = 0. \tag{4.15}$$

Therefore $r = -6$. Using this in (2.11), we obtain

$$S(X,Y) = -2g(X,Y). \tag{4.16}$$

Using (4.16) in (2.10), we have

$$R(X,Y)Z = -\{g(X,Z)Y - g(Y,Z)X\}. \tag{4.17}$$

i.e., the space has a constant sectional curvature -1. □

5. Conclusions

This research focuses on RB-solitons on three-dimensional para-Sasakian manifolds. We begin by considering the potential vector field V as affine conformal vector, leading to the Ricci tensor being a conformal quadratic Killing tensor. We also establish that potential vector field as affine conformal vector field is the Jacobi along the geodesic of ξ. Next, we investigate the case when the potential vector field V is conformal. We show that under this condition, the manifold is locally isometric to hyperbolic space, the vector field V is homothetic, and the manifold has a quadratic Killing tensor. We then consider the case where V is a torse-forming vector field. We prove that V is Jacobi along geodesics and we use ρ to analyze the nature of the soliton. Lastly, we demonstrate that for a gradient RB-soliton, the manifold has constant sectional curvature -1, which implies that it is locally isometric to hyperbolic space. In conclusion, this paper provides a thorough analysis of RB-solitons on three-dimensional para-Sasakian manifolds, examining different forms of the potential vector field and their geometric consequences.

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