DOI: 10.22098/jhs.2024.15582.1034

Journal of Hyperstructures 13 (2) (2024), 182-187

Journal Homepage: https://jhs.uma.ac.ir/

Research Paper

ON d-PRIME HYPERIDEALS OF HYPERRINGS

MARYAM AKHOUNDI[1,*] AND SABER OMIDI[2]

[1] Clinical Research Development Unit of Rouhani Hospital, Babol University of Medical Sciences, Babol, Iran, maryam.akhoundi@mubabol.ac.ir

[2] Department of Education, Ministry of Education Iran, Tehran, Iran, omidi.saber@yahoo.com

ARTICLE INFO

Article history:
Received: 05 August 2024
Accepted: 16 October 2024
Communicated by Ahmad Yousefian Darani

Keywords:
hyperring
hyperideal
homo-derivation

MSC:
16Y99; 13N15

ABSTRACT

For Krasner hyperrings, we study d-prime hyperideals where d is a homo-derivation. Furthermore, we show that every maximal d-hyperideal and d-prime hyperideal is a prime hyperideal of a commutative hyperring. Finally, we prove that if W is a d-prime hyperideal of a hyperring R and $d(q^n) \in W$ for some $q \in R$, then $d^2(q) \in W$.

1. Introduction

The notion of Krasner hyperrings was introduced by Krasner [10]. Marty [11] proposed the idea of the hypergroup in 1934. Heidari and Davvaz [7] proposed hyperideals for ordered semihypergroups in 2011. In [4, 20, 17, 19, 6, 15, 18], several new concepts and results of ordered hyperstructures are described.

Derivation in rings was first explored by Posner [14] and later (2013) on hyperrings by Asokkumar [1] and some fundamental properties were investigated in [9]. The notion of derivations appeared on the ordered semihyperrings in [16]. Also, see [13, 5, 3, 2, 8].

∗Address correspondence to M. Akhoundi, Clinical Research Development Unit of Rouhani Hospital, Babol University of Medical Sciences, Babol, Iran, maryam.akhoundi@mubabol.ac.ir.

In this note, we establish some theorems on d-prime hyperideals and obtain several results about homo-derivations on hyperrings.

Definition 1.1. The triple $(R, \oplus, \odot)$ is a *hyperring* if

(1) $(R, \oplus)$ is a canonical hypergroup [12];
(2) $(R, \odot)$ is a semigroup and $c \odot 0 = 0 = 0 \odot c$, $\forall c \in R$;
(3) The multiplication $\odot$ is distributive with respect to the hyperaddition $\oplus$.

A subset $\emptyset \neq K \subseteq R$ is a *hyperideal* if

(1) $(K, \oplus)$ is a canonical subhypergroup;
(2) for every $k \in K$ and $c \in R$, $c \odot k, k \odot c \in K$.

Definition 1.2. Let $(R, \oplus, \odot)$ be a hyperring. A function $d : R \to R$ is a *derivation* [9] if $\forall c, c' \in R$,

(1) $d(c \oplus c') \subseteq d(c) \oplus d(c')$;
(2) $d(c \odot c') \in d(c) \odot c' \oplus c \odot d(c')$,

Example 1.3. Consider the hyperring $R = \{0, 1, -1\}$:

$\oplus$	0	1	-1
0	0	1	-1
1	1	1	R
-1	-1	R	-1

$\odot$	0	1	-1
0	0	0	0
1	0	1	-1
-1	0	-1	1

Define $d : R \to R$ by

$$d(x) = \begin{cases} 0, & x = 0 \\ -1, & x = 1 \\ 1, & x = -1 \end{cases}$$

Then, d is a derivation on R.

2. d-PRIME HYPERIDEALS

Definition 2.1. Let $(R, \oplus, \odot)$ be a hyperring. A function $\Lambda : R \to R$ is a *homomorphism* if $\forall c, c' \in R$,

(1) $\Lambda(c \oplus c') \subseteq \Lambda(c) \oplus \Lambda(c')$;
(2) $\Lambda(c \odot c') = \Lambda(c) \odot' \Lambda(c')$.

A derivation d is called a *homo-derivation* if

$$d(c \odot c') = d(c) \odot d(c')$$

Theorem 2.2. *Let d be a homo-derivation of R. If K is a subhyperring of R, then*

$$d^{-1}(K) = \{x \in R \mid d(x) \in K\}$$

is a subhyperring of R.

Proof. For any $c, c' \in d^{-1}(K)$, we get $d(c), d(c') \in K$. So,

$$d(c \ominus c') \subseteq d(c) \ominus d(c') \subseteq K.$$

Thus, $c \ominus c' \subseteq d^{-1}(K)$. On the other hand,

$$d(c \odot c') = d(c) \odot d(c') \in K.$$

Hence, $c \odot c' \in d^{-1}(K)$. □

Definition 2.3. Let d be a homo-derivation of R. A hyperideal K of R, such that $K \neq R$, is called a *d-prime hyperideal* if

$$c \odot c' \in K \Rightarrow c \in K \text{ or } d(c') \in K, \forall c, c' \in R.$$

Remark 2.4. [8] For a hyperideal K of R,

$$\sqrt{K} := \{x \in R \mid \exists n \in \mathbb{N} \text{ such that } x^n \in K\}$$

is a hyperideal of R.

Theorem 2.5. *Let d be a homo-derivation of R. If K is a d-prime hyperideal of R, then $\sqrt{K}$ is a d-prime hyperideal of R.*

Proof. Let $c \odot c' \in \sqrt{K}$ and $c \notin \sqrt{K}$ for $c, c' \in R$. We show that $d(c') \in \sqrt{K}$. Since $c \odot c' \in \sqrt{K}$, we get

$$(c \odot c')^n \in K \text{ for some } n \in \mathbb{N}.$$

So, $c^n \odot c'^n \in K$. Since K is a d-prime hyperideal and $c^n \notin K$, we have $d(c'^n) \in K$. So, $(d(c'))^n = d(c'^n) \in K$. Thus, $d(c') \in \sqrt{K}$. □

Example 2.6. Let $R = \{0, q, r, c\}$ and

$\oplus$	0	q	r	c
0	0	q	r	c
q	q	$\{0, r\}$	$\{q, c\}$	r
r	r	$\{q, c\}$	$\{0, r\}$	q
c	c	r	q	0

$\odot$	0	q	r	c
0	0	0	0	0
q	0	q	r	c
r	0	r	r	0
c	0	c	0	c

Clearly,

$$d(x) = \begin{cases} 0, & x = 0, c \\ \\ r, & x = q, r \end{cases}$$

is a homo-derivation on a hyperring $(R, \oplus, \odot)$. Clearly, $K_1 = \{0, r\}$ and $K_2 = \{0, c\}$ are d-prime hyperideals of R.

Definition 2.7. A hyperideal K of R is said to be a *d-hyperideal* if

$$d(x) \in K, \forall x \in K.$$

Theorem 2.8. *Let d be a homo-derivation of a commutative hyperring $(R, \oplus, \odot)$. If K is a maximal d-hyperideal and d-prime, then K is prime.*

Proof. Let $q \odot b \in K$ with $q \notin K$ and $b \notin K$ for $q, b \in R$. Let $x \in y \oplus z \subseteq K \oplus < q >$ for $y \in K$ and $z \in < q >$. Then

$$\begin{aligned} x \odot b &\in (y \oplus z) \odot b \\ &= (y \oplus (r \odot q)) \odot b \\ &= (y \odot b) \oplus (r \odot q \odot b) \\ &\subseteq (K \odot R) \oplus (R \odot K) \\ &\subseteq K \oplus K \\ &\subseteq K. \end{aligned}$$

where $r \in R$. So, $b \odot x \in K$ with $b \notin K$. Since K is d-prime, we get

$$d(x) \in K \subseteq K \oplus < q >.$$

As K is a maximal d-hyperideal, $K = K \oplus < q >$. Hence, $q \in K$, a contradiction. Therefore, K is a prime hyperideal of R. □

Theorem 2.9. *Let d be a homo-derivation of a hyperring $(R, \oplus, \odot)$. Then K is a d-prime hyperideal of R iff for any hyperideals V and W of R, $V \odot W \subseteq K$ implies $V \subseteq K$ or $d(W) \subseteq K$.*

Proof. $(\Rightarrow)$: Let K be a d-prime hyperideal of R, $V \odot W \subseteq K$ and $V \nsubseteq K$, where V, W are hyperideals of R. We prove that $d(W) \subseteq K$. As $V \nsubseteq K$, there exists $v \in V$ such that $v \notin K$. Take any $w \in W$. Then,

$$v \odot w \in V \odot W \subseteq K.$$

Since K is a d-prime hyperideal and $v \notin K$, we get $d(w) \in K$ for all $w \in W$. Thus, $d(W) \subseteq K$.

$(\Leftarrow)$: Suppose that $v \odot w \in K$ for some $v, w \in R$. Then $< v \odot w > \subseteq K$. So,

$$< v > \odot < w > \subseteq < v \odot w > \subseteq K.$$

Hence, $< v > \subseteq K$ or $d(< w >) \subseteq K$. Thus, $v \in K$ or $d(w) \in K$. Thus, K is a d-prime hyperideal. □

Theorem 2.10. *Let d be a homo-derivation of a hyperring $(R, \oplus, \odot, 0, 1)$. If K is a d-prime hyperideal and $d(q^n) \in K$ for some $q \in R$, then $d^2(q) \in K$.*

Proof. Let $q \in R$ and $d(q^n) \in K$. Then,

$$\begin{aligned}
& d(q^n) \in K \\
& \Rightarrow d(\underbrace{q \odot q \odot \cdots \odot q}_{\text{n-copies}}) \in K \\
& \Rightarrow \underbrace{d(q) \odot d(q) \odot \cdots \odot d(q)}_{\text{n-copies}} \in K \\
& \Rightarrow (d(q))^n \in K.
\end{aligned}$$

So,

$$(d(q))^{n-1} \odot d(q) \in K.$$

As K is a d-prime hyperideal,

$$(d(q))^{n-1} \in K \text{ or } d(d(q)) \in K.$$

Thus,

$$(d(q))^{n-1} \in K \text{ or } d^2(q) \in K.$$

If $(d(q))^{n-1} \in K$, then

$$(d(q))^{n-2} \odot d(q) \in K.$$

As K is a d-prime hyperideal,

$$(d(q))^{n-2} \in K \text{ or } d(d(q)) \in K.$$

By continuing this process, we get

$$d(q) \in K \text{ or } d(d(q)) \in K.$$

If $d(q) \in K$, then $1 \odot d(q) = d(q) \in K$. Since K is a d-prime hyperideal and $1 \notin K$, we obtain $d(d(q)) \in K$, i.e., $d^2(q) \in K$. □

3. Conclusions

This study was conducted to investigate the significant relationship between homo-derivations and prime hyperideals in hyperrings. We have shown that every maximal homo-derivation-hyperideal and homo-derivation-prime hyperideal is a prime hyperideal of a commutative hyperring R. Furthermore, we proved that if W is a d-prime hyperideal of R and $d(q^n) \in W$ for some $q \in R$, then $d^2(q) \in W$. In our future work, we study fuzzy d-prime hyperideals in hyperrings.

Acknowledgments

The authors wish to thank the anonymous referees for their suggestions and comments which significantly improved the presentation of results in this manuscript.

References

[1] A. Asokkumar, *Derivations in hyperrings and prime hyperrings*, Iran. J. Math. Sci. Inform., **8** (2013), 1-13.

[2] D. Boucher and F. Ulmer, *Linear codes using skew polynomials with automorphisms and derivations*, Des. Codes. Cryptogr., **70(3)** (2014), 405-431.

[3] L. Creedon and K. Hughes, *Derivations on group algebras with coding theory applications*, Finite Fields Appl., **56** (2019), 247-265.

[4] B. Davvaz, P. Corsini and T. Changphas, *Relationship between ordered semihypergroups and ordered semigroups by using pseudoorder*, European J. Combin., **44** (2015), 208-217.

[5] M. M. Ebrahimi and H. Pajoohesh, *Inner derivations and homo-derivations on l-rings*, Acta Math. Hungar., **100(1-2)** (2003), 157-165.

[6] Z. Gu and X. Tang, *Ordered regular equivalence relations on ordered semihypergroups*, J. Algebra, **450** (2016), 384-397.

[7] D. Heidari and B. Davvaz, *On ordered hyperstructures*, Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys., **73(2)** (2011), 85-96.

[8] J. Jun, *Algebraic geometry over hyperrings*, Adv. Math., **323** (2018), 142-192.

[9] L. Kamali Ardekani and B. Davvaz, *Some notes on differential hyperrings*, Iranian Journal of Science and Technology, **39(1)**(2015), 101-111.

[10] M. Krasner, *A class of hyperrings and hyperfields*, International J. Math. and Math. Sci., **6(2)** (1983), 307-312.

[11] F. Marty, *Sur une generalization de la notion de groupe*, in: Proceedings of the 8th Congress Math. Scandinaves, Stockholm, Sweden, (1934), 45-49.

[12] J. Mittas, *Hypergroups canoniques*, Mathematica Balkanica, (1972), 165-179.

[13] S. Omidi and B. Davvaz, *Fundamentals of derivations on (ordered) hyper(near)-rings*, Beitr Algebra Geom., **60(3)** (2019), 537-553.

[14] E. C. Posner, *Derivations in prime rings*, Proc. Amer. Math. Soc., **8** (1957), 1093-1100.

[15] Y. Rao, S. Kosari, H. Guan, M. Akhoundi and S. Omidi, *A short note on the left A-Γ-hyperideals in ordered Γ-semihypergroups*, AKCE International Journal of Graphs and Combinatorics, **19(1)** (2022), 49-53.

[16] Y. Rao, S. Kosari, Z. Shao and S. Omidi, *Some properties of derivations and m-k-hyperideals in ordered semihyperrings*, Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys., **83(3)** (2021), 87-96.

[17] Y. Rao, S. Kosari, Z. Shao, M Akhoundi and S. Omidi, *A study on A-$\boldsymbol{I}$-Γ-hyperideals and (m,n)-Γ-hyperfilters in ordered Γ-Semihypergroups*, Discrete Dyn. Nat. Soc., **2021** (2021), 10 pages.

[18] Z. Shao, X. Chen, S. Kosari and S. Omidi, *On some properties of right pure (bi-quasi-)hyperideals in ordered semihyperrings*, Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys., **83(4)** (2021), 95-104.

[19] J. Tang, X. Feng, B. Davvaz and X. Y. Xie, *A further study on ordered regular equivalence relations in ordered semihypergroups*, Open Math., **16** (2018), 168-184.

[20] N. Tipachot and B. Pibaljommee, *Fuzzy interior hyperideals in ordered semihypergroups*, Ital. J. Pure Appl. Math., **36** (2016), 859-870.