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Research Paper

FRACTIONAL ORDER PD$^{\alpha}$-TYPE ILC FOR LINEAR CONTINUOUS TIME-DELAY SWITCHED SYSTEM WITH DISTURBANCE MEASUREMENT AND UNCERTAINTIES NOISE

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ABSTRACT

This study investigates the efficacy of a novel PD$^{\alpha}$-type fractional-order iterative learning control (FOILC) approach for a class of fractional-order linear continuous-time delaying switched systems. The approach is evaluated in terms of L^p norm performance, aiming to mitigate the challenges associated with time delays in repetitive regulation of fractional-order linear systems. The generalized Young inequality of the convolution integral is used to leverage the resilience of the PD$^{\alpha}$-type approach in the iteration domain when the systems are perturbed by constrained external disturbances. We next analyze the convergence of the techniques for noise-free systems. The results demonstrate that it is feasible to guarantee both convergence and robustness over the duration of the experiment in certain situations. We study the convergence of error for the proposed class of fractional-order linear continuous-time delaying switched systems.

1. Introduction

Iterative learning control (ILC), among the most efficient intelligent control techniques, is Iteratively generating a series of deal with commands over a definite, finite time duration or interval, this invention dates back to the 1980s. ILC's main goal is to create an

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enhanced command and management as the following operation that takes into account any proportional, integral, or derivative tracking mistakes from the preceding operation. As the operation moves closer to infinity, the goal is for the successive ILC inputs, a mechanism to promote following a targeted trajectory that is as flawless as humanly feasible.

The fractional-order ILC (FOILC) schemes have emerged in recent years. For this reason, the observation of fractional-order systems as indicated above. With the use of fractional calculus, many real-world energizing systems, like colored noise, may be precisely represented. The switching signal could be state-driven or time-driven; it might be a member of a particular set with a variety of members, and it might be diverse. A switching sequence, a limited number of discrete-time or continuous-time subsystems, and hybrid systems are called switched systems. The switching sequence, which determines which subsystem is enabled for a brief period of time, is typically random. Switched systems are widely used in engineering for a variety of purposes, including the management of chemical processes, power systems, process control, traffic systems, and many more sectors [21]-[19].

In 2001, Chen et al. [4] studied the frequency domain and proposed a fractional-order D-type ILC algorithm, examining its convergence using the recursively direct discretization method. However, according to FOILC guidelines, one issue with the existing convergence criteria is that it does not guarantee that the true tracking error monotonically decreases. This suggests that, although the iteration-wise tracking error may sporadically approach engineering precision, it is not always assured to decrease monotonically because the FOILC updating method terminates after a finite number of operations (see [7]-[12] for details).

In 2013, Bu et al. [3] studied an assortment of linear discrete-time switched systems with unspecified switching signals. It is expected that the switched system is repeatedly operated over a finite time period, after which a first-order P-type ILC method can be utilized to provide flawless tracking throughout the entire time interval.

In 2013, Lan et al. examined fractional-order D^{α}-type ILC for non-linear time-delay systems and developed the convergence criteria. In 2014, Xuan et al. [23] utilized the super vector technique to discuss convergence under noise-free conditions and analyzed resilience under limited noise disturbance of a controlled system. In 2016, Lan et al. [9] analyzed the ILC design challenge, which is transformed into a stability problem for a discrete system by understanding the mechanisms of control and instruction, and created a discrete system for P-type fractional-order ILC.

The accompanying theorem and its proof establish the necessary conditions for the convergence of the proposed PD-alpha type ILC in the time domain, applicable to a class of fractional-order singular systems. Independently, Chenchen and Jing [5] derived the convergence requirements for closed-loop PD^{α}-type ILC in fractional-order systems with nonlinear time delay. In 2018, Yan et al. [15] investigated PD^{α}-type ILC for fractional delay systems. Meanwhile, Lazarevic et al. [14] examined state-space forms of fractional-order linear singular time-delay systems. Most recently, in 2023, Dewangan [6] conducted a convergence analysis of proportional-derivative-type ILC for linear continuous constant time delay switched systems with observation noise and state uncertainties, yielding significant results.

The remainder of this research paper is organized as follows: Section 2 provides an update on related mathematical definitions, preliminaries, and mathematical formulation. The main analysis of the proposed PD^{α}-type FOILC algorithm is presented in Section 3, which includes robustness and convergence results for fractional-order linear continuous-time delay switched systems with state uncertainties and measurement disturbances. Finally, Section 4 presents the conclusions.

2. Preliminaries

Firstly, we present key definitions and lemmas that are essential for proving our result and underpin the proposed methodology.

Definition 2.1. [8] Let $h : [0, T] \to \mathbb{R}^n$, be continuous time varying vector valued function defined by

$$h(t) = [h^1(t), h^2(t), \cdots, h^n(t)]^T,$$

its L^p-norm and λ-norm are given by

$$\begin{cases} \|h(t)\|_p &= \left[\int_0^T \left(\max_{1\le i\le n} |h^i(t)|\right)^p dt\right]^{1/p}, \quad 1 \le p \le \infty. \\ \|h(t)\|_\lambda &= \sup_{0\le t\le T} e^{-\lambda t} \left(\max_{1\le i\le n}(|h^i(t)|)\right), \quad \lambda > 0, \end{cases}$$

Definition 2.2. [18] For a function f, The Riemann-Liouville's fractional integral of order $\alpha > 0$ is defined as

$${}_{t_0}D_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^t (t-s)^{\alpha-1} f(s) ds,$$

where $\Gamma(\alpha)$ is the Gamma function.

Lemma 2.3. [20] *Let $g(t) \in L^q$ and $h(t) \in L^p$, $t \in [0, T]$, be Lebesgue integrable functions. If the convolution integral of g and h exists, then*

$$(g * h)(t) = \int_0^T g(t-s)h(s)ds.$$

The generalized Young inequality for the convolution integral is

$$\|(g * h)(t)\|_r \le \|g(t)\|_q \|h(t)\|_p,$$

where $1 \le p, q, r \le \infty$ and $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$. In particular, when $r = p$, the inequality becomes

$$\|(g * h)(t)\|_p \le \|g(t)\|_1 \|h(t)\|_p.$$

Due to its similarity in form to integer-order differential equations in initial conditions, the Caputo formulation is most frequently employed in engineering. Consequently, the Caputo fractional definition was selected as the primary instrument for investigation. The definition of the function was comparable to that in [22].

$${}_{t_0}^{C}D_t^{\alpha} f(t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_{t_0}^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}} d\tau, & \text{if } n-1 < \alpha < n, \\ \frac{d^n f(t)}{dt^n}, & \alpha = n \end{cases}$$

where t_0 is the starting time, ${}_{t_0}^{C}D_t$ represents a fractional-order integral operator on $[t_0, t]$, and $\Gamma(\cdot)$ denotes the Gamma function. Specifically, if $0 < \alpha < 1$, then

$${}_{t_0}^{C}D_t f(t) = \frac{1}{\Gamma(n-\alpha)} \int_{t_0}^t \frac{f'(\tau)}{(t-\tau)^{\alpha}} d\tau.$$

Here, $\alpha \in \mathbb{R}^+$, and $[\alpha]$ indicates the integral part of α."

Lemma 2.4. [7] *Consider a continuous function $f(u(t),t)$. Then, the initial value problem*

$$\begin{cases} {}_{t_0}^{C}D_t^{\alpha}u(t) & = f(u(t),t), \\ u(t_0) & = u_0, \end{cases}$$

where $0 < \alpha < 1$, is equivalent to the following nonlinear Volterra integral equation:

$$u(t) = u_0 + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} f(u(s),s)ds.$$

3. Problem statement and analysis

Consider the fractional-order linear continuous-time delay switching system with the following state uncertainties and measurement disturbances:

$$\begin{cases} {}_{0}^{C}D_t^{\alpha}x_k(t) & = A_{\sigma(i)}x_k(t) + D_{\sigma(i)}x_k(t-\tau) + B_{\sigma(i)}u_k(t) + \xi_k(t), \\ y_k(t) & = C_{\sigma(i)}x_k(t) + \eta_k(t), \end{cases} \tag{3.1}$$

where $t \in [0,T]$, and

(a) $\alpha \in (0,1)$, and $\cdot^{(\alpha)}$ denotes the Caputo derivative. ${}_{0}^{C}D_t^{\alpha}$: $x_k(t) \in \mathbb{R}^n$ is the state vector for the k that belongs to the set of natural numbers.
(b) $u_k(t) \in \mathbb{R}^p$ is the input vector.
(c) $y_k(t) \in \mathbb{R}^q$ is the output vector.
(d) $A_{\sigma(t)} \in \mathbb{R}^{n\times n}$, $B_{\sigma(t)} \in \mathbb{R}^{n\times p}$, $C_{\sigma(t)} \in \mathbb{R}^{q\times n}$, and $D_{\sigma(t)} \in \mathbb{R}^{n\times n}$ are the state matrices, input matrices, output matrices, and state matrices with delay term, respectively.
(e) $\sigma(t)$ is defined as $\sigma(t) : [0,T] \to G = \{1,2,3,\cdots,m\}$, where m is the number of subsystems in the switched system. Alternatively, the matrices group $(A_{\sigma(t)}, B_{\sigma(t)}, C_{\sigma(t)}, D_{\sigma(t)})$ is a random component of the following set

$$\{(A_1,B_1,C_1,D_1),(A_2,B_2,C_2,D_2),\cdots,(A_m,B_m,C_m,D_m)\}$$

(f) $\xi_k(t) \in \mathbb{R}^n$ and $\eta_k \in \mathbb{R}^n$ are bounded state disturbance and bounded measurement noise, respectively, with $\|\xi_k(t)\|p \le b_\xi$, $\|\eta_k(t)\|p \le b_\eta$, and $\|\eta_k^{(\alpha)}(t)\|p \le b_{\eta\alpha}$, where b_ξ, b_η, and $b_{\eta\alpha}$ are positive constants. Here $\eta_k^{(\alpha)}(t)$ denotes the derivative of $\eta_k(t)$ of fractional order $\alpha \in (0,1)$ with respect to t.

Here are some fundamental presumptions of the switching system:
Assumption 1 System (3.1) can operate respectively on the limited time period $[0,T]$, and the initial state can be reset for all iterations, that is,

$$x_k(0) = x_d(0), \quad k = 1,2,3,4,5,6,...$$

where the desired state's starting value is $x_d(0)$.
Assumption 2 The desired outcome, $y_d(t)$, is specified and remains invariant throughout the iteration process.
Assumption 3 The switching sequence $\sigma(t)$ remains iteration-invariant once it is randomly specified at the initial iteration.
Assumption 4 Only the intended control inputs, $u_d(t)$ and $x_d(t)$, exist for the specified

desired output $y_d(t)$, such that

$$\begin{cases} {}_0^C D_t^{\alpha} x_d(t) & = A_{\sigma(i)} x_d(t) + D_{\sigma(i)} x_d(t-\tau) + B_{\sigma(i)} u_d(t) + \xi_d(t), \\ y_d(t) & = C_{\sigma(i)} x_d(t). \end{cases}$$

Here, τ denotes the delay time, such that each subsystem's stay time is larger than its delay time. Therefore

$$\tau < t_i - t_{i-1}, \quad \forall i \in M = \{1, 2, 3, \ldots, m\}.$$

Assumption 5 $C_i B_i, i = 1, 2, 3, 4, 5, ..., m$, are full rank matrices.

Assumption 6 For any t belonging to $[0, T]$, the value of $\Delta x_k(-t)$ is always zero.

We regard the PD^{α}-type FOILC method in this article as

$$u_{k+1}(t) = u_k(t) + \Gamma_p e_k(t) + \Gamma_d e_k^{(\alpha)}(t), \tag{3.2}$$

where $\Gamma_p \in \mathbb{R}^{p\times q}$ and $\Gamma_d \in \mathbb{R}^{p\times q}$ are, respectively, the gain learning matrix and the proportional learning matrix, and

$$e_{k+1}(t) = y_d(t) - y_{k+1}(t),$$

represents the tracking error at the $(k+1)^{th}$ iteration, where $y_d(t)$ denotes the system's expected, targeted, or desired output. The following fundamental switching system assumptions are possible.

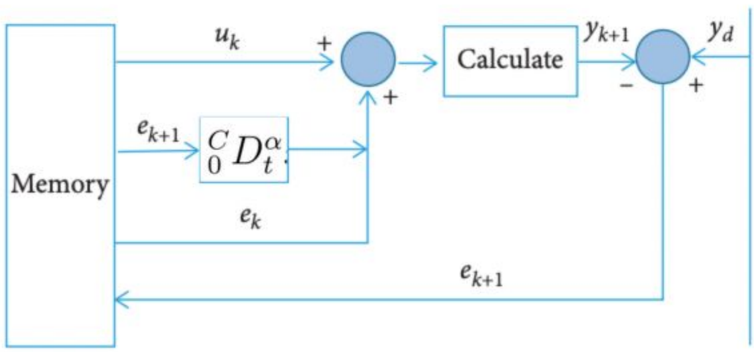

FIGURE 1. Block diagram of the open- and closed-loop PD$^{\alpha}$-type ILC algorithm

For the sake of simplicity, let's assume each subsystem operates only once during the time interval $[0, T]$, and the switching rule $\sigma(t)$ can be characterised as

$$\sigma(t) = i = \begin{cases} 1, & t \ \text{belong to} \ [0, t_1), \\ 2, & t \ \text{belong to} \ [t_1, t_2), \\ \vdots & \\ m, & t \ \text{belong to} \ [t_{m-1}, T]. \end{cases} \tag{3.3}$$

The sequence suggests that each subsystem is activated only once throughout the interval $[0, T]$.

As a result, the system (3.1) with the m subsystem is rewritten as

$$\begin{cases} {}_0^C D_t^\alpha x_k(t) & = A_i x_k(t) + D_i x_k(t-t_M) + B_i u_k(t) + \xi_k(t), \\ y_k(t) & = C_i x_k(t) + \eta_k(t), \end{cases} \tag{3.4}$$

where i belongs to the set $\{1, 2, ..., m\}$. Considering the initial time $t_0 = 0$ and the final time $t_m = T$, and in accordance with Lemma 2.4 and Equation 3.4, the system is in the following state:

$$\begin{aligned} x_k(t) = x_k(t_{i-1}) &+ \frac{1}{\Gamma(\alpha)} \int_{t_{i-1}}^{t} (t-s)^{\alpha-1} A_i x_k(s) ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_{t_{i-1}}^{t} (t-s)^{\alpha-1} D_i x_k(t-\tau) ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_{t_{i-1}}^{t} (t-s)^{\alpha-1} B_i u_k(s) ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_{t_{i-1}}^{t} (t-s)^{\alpha-1} \xi_k(s) ds, \ t \in [t_{i-1}, t_i], 1 \le i \le m. \end{aligned} \tag{3.5}$$

Remark 3.1. The initial conditions specified in Assumption 1 are applicable only to the first subsystem, but not to the subsequent subsystems; specifically, the condition $x_k(t_i) \neq x_d(t_i)$ holds for $i = 1, 2, ..., m$.

4. Main results

Now, we discuss the asymptotic convergence of the proposed system.

Theorem 4.1. *Assume the system (3.4) satisfied the assumption* $1-6$ *when control law (3.2) is applied to the system (3.4); if*

(i) $M_i = (\Gamma(\alpha) - \|(t)^{\alpha-1} A_i\|_1 - \|(t)^{\alpha-1} D_i\|_1) > 0, \quad i = 1, 2, 3, ..., m.$

(ii) $\rho_i = (\|I - \Gamma_d C_i B_i\| + \beta_i) < 1$, *where*

$$\beta_i = \frac{(\|\Gamma_p C_i + \gamma_p C_i A_i\| + \|\Gamma_d C_i D_i\|) \|(t)^{\alpha-1} B_i\|_1}{M_i},$$

(iii) $\gamma_i = \frac{(\|\Gamma_p C_i + \Gamma_d C_i A_i\|) \|(t)^{\alpha-1}\|_1}{M_i} + \|\Gamma_d C_i\|, i \in G = \{1, 2, 3, ...m\}.$

Then, the tracking error is bounded uniformly, and during the range $[0, T]$ *as* $k \to \infty$, *the system's output asymptotically falls into a tiny neighbourhood of the targeted or desired output.*

Proof. We know that $\Delta x_k(t)$ and $\Delta u_k(t)$ at a time t are defined as follows:

$$\Delta x_k(t) = x_d(t) - x_k(t),$$

and

$$\Delta u_k(t) = u_d(t) - u_k(t),$$

and $y_k^{(\alpha)}(t)$ and $y_d^{(\alpha)}(t)$ can be formulate as follows:

$$\begin{aligned} y_k^{(\alpha)}(t) &= C_i x_k^{(\alpha)}(t) + \eta_k^{(\alpha)}(t), \\ y_d^{(\alpha)}(t) &= C_i x_d^{(\alpha)}(t). \end{aligned}$$

Using the concept of tracking error, we have the following

$$\begin{aligned}
e_k^{(\alpha)}(t) &= y_d^{(\alpha)}(t) - y_k^{(\alpha)}(t) \\
&= C_i x_d^{(\alpha)}(t) - C_i x_k^{(\alpha)}(t) - \eta_k^{(\alpha)}(t) \\
&= C_i(x_d^{(\alpha)}(t) - x_k^{(\alpha)}(t)) - \eta_k^{(\alpha)}(t) \\
&= C_i \Delta x_k^{(\alpha)}(t) - \eta_k^{(\alpha)}(t) \\
&= C_i[A_i \Delta x_k(t) + D_i \Delta x_k(t-\tau) + B_i \Delta u_k(t) - \xi_k(t)] - \eta_k^{(\alpha)}(t) \\
&= C_i A_i \Delta x_k(t) + C_i D_i \Delta x_k(t-\tau) \\
&\quad + C_i B_i \Delta u_k(t) - C_i \xi_k(t) - \eta_k^{(\alpha)}(t).
\end{aligned} \tag{4.1}$$

Now, from using equation (3.2), we get as follows:

$$\begin{aligned}
\Delta u_{k+1}(t) &= u_d(t) - u_{k+1}(t) \\
&= u_d(t) - u_k(t) - \Gamma_p e_k(t) - \Gamma_d e_k^{(\alpha)}(t) \\
&= \Delta u_k(t) - \Gamma_p e_k(t) - \Gamma_d e_k^{(\alpha)}(t),
\end{aligned} \tag{4.2}$$

Putting the value of (4.1) into (4.2), we can write as follows:

$$\begin{aligned}
\Delta u_{k+1}(t) &= \Delta u_k(t) - \Gamma_p(C_i \Delta x_k(t) - \eta_k(t)) \\
&\quad - \Gamma_d[C_i A_i \Delta x_k(t) + C_i D_i \Delta x_k(t - t_M) \\
&\quad + C_i B_i \Delta u_k(t) - C_i \xi_k(t) - \eta_k^{(\alpha)}(t)], \\
&= (I - \Gamma_d C_i B_i)\Delta u_k(t) - (\Gamma_p C_i + \Gamma_d C_i A_i)\Delta x_k(t) \\
&\quad - \Gamma_d C_i D_i \Delta x_k(t - t_M) + \Gamma_d C_i \xi_k(t) + \Gamma_d \eta_k^{(\alpha)}(t) + \Gamma_p \eta_k(t)
\end{aligned} \tag{4.3}$$

where $t \in [t_{i-1}, t_i]$, the L^p norm of the function f defined by $f[t_{i-1}, t_i], t \in [0, t_m]$, and $t \in [t_{i-1}, t_i] \subset [0, T] \implies \|f[t_i, t_{i+1}]\|_p \le \|f\|_p$.

Step 1: Assume that t firstly lies in the first sub-interval, i.e., $t \in [0, t_1]$. In this case, the first subsystem is active. The system's state can be represented by the following equation (3.4):

$$\begin{aligned}
x_{k+1}(t) &= x_{k+1}(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} A_1 x_{k+1}(s) ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} D_1 x_{k+1}(t - t_M) ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} B_1 u_{k+1}(s) ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \xi_{k+1}(s) ds.
\end{aligned} \tag{4.4}$$

Under the assumption stated in Assumption 1, we derive

$$\begin{aligned}\Delta x_{k+1}(t) &= x_d(t) - x_{k+1}(t)\\ &= x_d(t) - x_{k+1}(0) - \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} A_1 x_{k+1}(s)ds\\ &- \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} D_1 x_{k+1}(s-\tau)ds\\ &- \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} B_1 u_{k+1}(s)ds\\ &- \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} \xi_{k+1}(s)d.\end{aligned}$$

Consequently, we obtain:

$$\begin{aligned}\Delta x_{k+1}(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} A_1 \Delta x_{k+1}(s)ds\\ + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} D_1 \Delta x_{k+1}(s-\tau)ds\\ + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} B_1 \Delta u_{k+1}(s)ds\\ + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} \xi_{k+1}(s)ds.\end{aligned} \tag{4.5}$$

Now, we observe that, for the case when $0 < t < \tau$, we get

$$\begin{aligned}\int_0^t \|\Delta x_{k+1}(s-\tau)\|ds &= \int_{-\tau}^{t-\tau} \|\Delta x_{k+1}(s)\|ds\\ &\leq \int_{-\tau}^{0} \|\Delta x_{k+1}(s)\|ds.\end{aligned} \tag{4.6}$$

and, when $t_M < t < t_1$, we have

$$\begin{aligned}\int_0^t \|\Delta x_{k+1}(s-\tau)\|ds &= \int_{-\tau}^{t-\tau} \|\Delta x_{k+1}(s)\|ds\\ &= \int_{-\tau}^{0} \|\Delta x_{k+1}(s)\|ds + \int_0^{t-\tau} \|\Delta x_{k+1}(s)\|ds\\ &\leq \int_{-\tau}^{0} \|\Delta x_{k+1}(s)\|ds + \int_0^{t} \|\Delta x_{k+1}(s)\|ds,\end{aligned} \tag{4.7}$$

and assumption 6 gives as

$$\int_{-\tau}^{0} \|\Delta x_{k+1}(s)\|ds = 0, \tag{4.8}$$

From (4.6),(4.7) and (4.8), we derive

$$\int_0^t \|\Delta x_{k+1}(s-\tau)\|ds \leq \int_0^t \|\Delta x_{k+1}(s)\|.ds$$

Similarly, we can obtain

$$\int_0^T \|(t-s)^{(\alpha-1)}D_1\|_1\|\Delta x_{k+1}(s-\tau)\|_p ds \leq \int_0^T \|(t-s)^{(\alpha-1)}D_1\|_1\|\Delta x_{k+1}(s)\|_p ds,$$

that is,

$$\|\Delta x_{k+1}(t-\tau)\|_p \leq \|\Delta x_{k+1}(t)\|_p.$$

Taking the L^p norm over the interval $t \in [0, t_1]$ on both sides of the above inequality, we get:

$$\|\Delta x_{k+1}[0, t_1-\tau]\|_p \leq \|\Delta x_{k+1}[0, t_1]\|_p.$$

Applying the generalised Young inequality to the convolution integral on both sides of (4.5), we obtain

$$\begin{aligned}
\|\Delta x_{k+1}[0,t_1]\|_p &= \left[\int_0^T \max|x^i_{k+1}(t)|^p\right]^{\frac{1}{p}} \\
&\leq \frac{1}{\Gamma(\alpha)}\|(t-s)^{(\alpha-1)}A_1\|_1\|\Delta x_{k+1}[0,t_1]\|_p \\
&+ \frac{1}{\Gamma(\alpha)}\int_0^T \|(t-s)^{(\alpha-1)}D_1\|_1\|\Delta x_{k+1}[0,t-t_M]\|_p ds \\
&+ \frac{1}{\Gamma(\alpha)}\|(t)^{\alpha-1}B_1\|_1\|\Delta u_{k+1}[0,t_1]\| + \frac{1}{\Gamma(\alpha)}\|(t)^{\alpha-1}\|_1 b_\xi \\
&\leq \frac{1}{\Gamma(\alpha)}\|(t)^{\alpha-1}A_1\|_1\|\Delta x_{k+1}[0,t_1]\|_p \\
&+ \frac{1}{\Gamma(\alpha)}\|(t)^{\alpha-1}D_1\|_1\|\Delta x_{k+1}[0,t_1-\tau]\|_p \\
&+ \frac{1}{\Gamma(\alpha)}\|(t)^{\alpha-1}B_1\|_1\|\Delta u_{k+1}[0,t_1]\| + \frac{1}{\Gamma(\alpha)}\|(t)^{\alpha-1}\|_1 b_\xi \\
&\leq \frac{1}{\Gamma(\alpha)}\|(t)^{\alpha-1}A_1\|_1\|\Delta x_{k+1}[0,t_1]\|_p \\
&+ \frac{1}{\Gamma(\alpha)}\|(t)^{\alpha-1}D_1\|_1\|\Delta x_{k+1}[0,t_1]\|_p \\
&+ \frac{1}{\Gamma(\alpha)}\|(t)^{\alpha-1}B_1\|_1\|\Delta u_{k+1}[0,t_1]\| + \frac{1}{\Gamma(\alpha)}\|(t)^{\alpha-1}\|_1 b_\xi.
\end{aligned} \tag{4.9}$$

So, we have

$$\begin{aligned}
\|\Delta x_{k+1}[0,t_1]\|_p &\leq \frac{\|(t)^{\alpha-1}B_1\|_1\|\Delta u_{k+1}[0,t_1]\|_p + \|(t)^{\alpha-1}\|_1 b_\xi}{\Gamma(\alpha) - \|(t)^{\alpha-1}A_1\|_1 - \|(t)^{(\alpha-1)}D_1\|} \\
&\leq \frac{\|(t)^{\alpha-1}B_1\|_1\|\Delta u_{k+1}[0,t_1]\|_p + \|(t)^{\alpha-1}\|_1 b_\xi}{M_1},
\end{aligned} \tag{4.10}$$

where $M_1 = (\Gamma(\alpha) - \|(t)^{\alpha-1}A_1\|_1 - \|(t)^{(\alpha-1)}D_1\|_1) > 0$.
From (4.3), we have

$$\begin{aligned}
\Delta u_{k+1}(t) &= (I - \Gamma_d C_1 B_1)\Delta u_k(t) - (\Gamma_d C_1 + \Gamma_d C_1 A_1)\Delta x_k(t) \\
&- \Gamma_d C_1 D_1 \Delta x_k(t-\tau) + \Gamma_d C_1 \xi_k(t) + \Gamma_p \eta_k(t) + \Gamma_d \eta_k^{(\alpha)}(t).
\end{aligned} \tag{4.11}$$

Computing the L^p norm over the interval $t \in [0, t_1]$ on both sides of (4.11), we observe that

$$\begin{aligned}\|\Delta u_{k+1}[0,t_1]\| \leq{}& \|I - \Gamma_d C_1 B_1\|_1 \|\Delta u_k[0,t_1]\|_p \\ &+ \|\Gamma_p C_1 + \Gamma_d C_1 A_1\|_1 \|\Delta x_k[0,t_1]\|_p \\ &+ \|\Gamma_d C_1 D_1\|_1 \|\Delta x_k[0, t_1 - t_M]\|_p + \|\Gamma_d C_1\| b_\xi \\ &+ \|\Gamma_p\| b_\eta + \|\Gamma_d\| b_{\eta\alpha},\end{aligned} \tag{4.12}$$

Substituting (4.10) into (4.12) yields

$$\begin{aligned}\|\Delta u_{k+1}[0,t_1]\|_1 \leq{}& (\|I - \Gamma_d C_1 B_1\|) \|\Delta u_k[0,t_1]\|_p \\ &+ (\|\Gamma_p C_1 + \Gamma_d C_1 A_1\| + \|\Gamma_d C_1 D_1\|) \\ &\left[\frac{[\|(t)^{\alpha-1} B_1\| \|\Delta u_{k+1}[0,t_1]\|_p + \|(t)^{\alpha-1}\| b_\xi]}{M_1}\right] \\ &+ \|\Gamma_d C_1\| b_\xi + \|\Gamma_p\| b_\eta + \|\Gamma_d\| b_{\eta\alpha} \\ \leq{}& \left(\|I - \Gamma_d C_1 B_1\|_1 + \frac{(\|\Gamma_p C_1 + \Gamma_d C_1 A_1\|_1 + \|\Gamma_d C_1 D_1\|)\|(t)^{\alpha-1} B_1\|}{M_1}\right) \\ &\|\Delta u_{k+1}[0,t_1]\|_p \\ &+ \left(\frac{[\|\Gamma_p C_1 + \Gamma_d C_1 A_1\|_1 + \|\Gamma_d C_1 D_1\|]\|(t)^{\alpha-1}\|}{M_1} + \|\Gamma_d C_1\|\right) b_\xi \\ &+ \|\Gamma_p\| b_\eta + \|\Gamma_d\| b_{\eta\alpha} \\ \leq{}& (\|I - \Gamma_d C_1 B_1\| + \beta_1) \|\Delta u_{k+1}[0,t_1]\|_p + \gamma_1 b_\xi + \|\Gamma_p\| b_\eta + \|\Gamma_d\| b_{\eta\alpha},\end{aligned} \tag{4.13}$$

where

$$\beta_1 = \frac{(\|\Gamma_p C_1 + \Gamma_d C_1 A_1\| + \|\Gamma_d C_1 D_1\|)\|(t)^{\alpha-1} B_1\|}{M_1},$$
$$\gamma_1 = \frac{[\|\Gamma_p C_1 + \Gamma_d C_1 A_1\| + \|\Gamma_d C_1 D_1\|]\|(t)^{\alpha-1}\|}{M_1} + \|\Gamma_d C_1\|_1,$$

and we have

$$\begin{aligned}\|\Delta u_{k+1}[0,t_1]\|_p \leq{}& (\|I - \Gamma_d C_1 B_1\| + \beta_1) \|\Delta u_k[0,t_1]\|_p + \gamma_1 b_\xi + \\ &+ \|\Gamma_p\| b_\eta + \|\Gamma_d\| b_{\eta\alpha} \\ ={}& \rho_1 \|\Delta u_k[0,t_1]\|_p + \gamma_1 b_\xi + \|\Gamma_p\| b_\eta + \|\Gamma_d\| b_{\eta\alpha},\end{aligned} \tag{4.14}$$

where $\rho_1 = (\|I - \Gamma_d C_1 B_1\| + \beta_1)$. using (4.14), we have

$$\begin{aligned}\|\Delta u_k[0,t_1]\| \leq{}& \rho_1^k \|\Delta u_1[0,t_1]\|_p - \frac{\rho_1^k}{1-\rho}(\gamma_1 b_\xi + \|\Gamma_p\| b_\eta + \|\Gamma_d\| b_{\eta\alpha}) \\ &+ \frac{\gamma_1 b_\xi + \|\Gamma_p\| b_\eta + \|\Gamma_d\| b_{\eta\alpha}}{1-\rho_1}.\end{aligned} \tag{4.15}$$

Utilising conditions (ii) and (iii) of Theorem 4.1 and taking the supremum over $t \in [0, t_1]$, we derive

$$\lim_{k\to\infty} \sup_{0 \leq t \leq t_1} \|\Delta u_k[0,t_1]\|_p \leq \frac{\gamma_1 b_\xi + \|\Gamma_p\| b_\eta + \|\Gamma_d\| b_{\eta\alpha}}{1-\rho}. \tag{4.16}$$

Combining (4.10) and (4.16), we have

$$\begin{aligned}\lim_{k\to\infty}\sup_{0\le t\le t_1}&\|\Delta x_{k+1}[0,t_1]\|_p\\&\le\frac{\|(t)^{\alpha-1}B_1\|\|\Delta u_{k+1}[0,t_1]\|_p++\|(t)^{\alpha-1}\|_1 b_\xi}{M_1}\\&\le\|(t)^{\alpha-1}B_1\|\frac{[\gamma_1 b_\xi+\|\Gamma_p\|b_\eta+\|\Gamma_d\|b_{\eta\alpha}]}{M_1(1-\rho_1)}+\frac{\|(t)^{\alpha-1}\|b_\xi}{M_1}.\end{aligned}\tag{4.17}$$

Now, we observe that

$$\begin{aligned}e_{k+1}&=y_d(t)-y_k(t)\\&=C_1(x_d(t)-x_{k+1}(t))-\eta_{k+1}(t)\\&=C_1\Delta x_{k+1}(t)-\eta_{k+1}(t).\end{aligned}\tag{4.18}$$

Taking the L^p norm on $t\in[0,t_1]$ on both sides of equation (4.18) and computing the supremum, substituting (4.17) into (4.18), we derive

$$\begin{aligned}\lim_{k\to\infty}\sup_{0\le t\le t_1}&\|e_{k+1}[0,t_1]\|_p\\&\le\|C_1\|\left(\frac{\|(t)^{\alpha-1}B_1\|\gamma_1}{M_1(1-\rho_1)}+\frac{\|(t)^{\alpha-1}\|}{M_1}\right)b_\xi\\&\quad+\left(\frac{\|C_1\|\|(t)^{\alpha-1}B_1\|\|\Gamma_p\|}{M_1(1-\rho_1)}+1\right)b_\eta\\&\quad+\left(\frac{\|C_1\|\|(t)^{\alpha-1}B_1\|\|\Gamma_d\|_1}{M_1(1-\rho_1)}\right)b_{\eta\alpha}\\&\quad+\|C_1\|\|(t)^{\alpha-1}D_1\|\|\Delta x_{k+1}[0,t_1-t_M]\|_p.\end{aligned}\tag{4.19}$$

this implies

$$\|\Delta x_{k+1}[0,t_1]\|\le\frac{\|(t)^{\alpha-1}B_1\|\|\Delta u_{k+1}[0,t_1]\|_p+\|(t)^{(\alpha-1)}\|b_\xi}{M_1(1-\|(t)^{\alpha-1}\|)}.\tag{4.20}$$

From (4.20) and (4.16), we have

$$\begin{aligned}\lim_{k\to\infty}\sup_{0\le t\le t_1}\|\Delta x_{k+1}[0,t_1]\|_p\le&\frac{\|(t)^{\alpha-1}B_1\|(\gamma_1 b_\xi+\|\Gamma_p\|b_\eta+\|\Gamma_d\|b_{\eta\alpha})}{M_1(1-\rho)}\\&+\frac{\|(t)^{\alpha-1}\|b_\xi}{M_1}\end{aligned}\tag{4.21}$$

According to (3.4), we have

$$\begin{aligned}e_{k+1}&=y_d(t)-y_k(t)\\&=C_1(x_d(t)-x_{k+1})-\eta_{k+1}(t)\\&=C_1\Delta x_{k+1}(t)-\eta_{k+1}(t).\end{aligned}\tag{4.22}$$

Taking supremum L^p norm $t \in [0, t_1]$, we obtain

$$
\begin{aligned}
\lim_{k\to\infty} \sup_{0\le t\le t_1} &\|\Delta e_{k+1}[0,t_1]\|_p \\
&\le \|C_1\| \left(\|(t)^{\alpha-1} B_1\| \frac{[\gamma_1 b_\xi + \|\Gamma_p\| b_\eta + \|\Gamma_d\| b_{\eta\alpha}]}{1-\rho} + \frac{\|(t)^{\alpha-1}\| b_\xi}{M_1} \right) \\
&\qquad + \|\eta_{k+1}[0,t_1]\| \\
&\le \left(\frac{\|C_1\|\|(t)^{\alpha-1} B_1\|\gamma_1}{M_1(1-\rho_1)} + \frac{\|C_1\|\|(t)^{\alpha-1}\|}{M_1} \right) b_\xi \\
&+ \left(\frac{\|C_1\|\|(t)^{\alpha-1} B_1\|\|\Gamma_p\|}{M_1(1-\rho)} + 1 \right) b_\eta + \left(\frac{\|C_1\|\|(t)^{\alpha-1} B_1\|\|\Gamma_d\|}{M_1(1-\rho_1)} \right) b_{\eta\alpha},
\end{aligned} \tag{4.23}
$$

Clearly, the RHS of equation (4.23) is bounded as $k \to \infty$; this shows that the tracking error of the subsystem (3.1) is bounded uniformly on the interval $[0, t_1]$.

Step 2: Let $t \in [t_1, t_2]$,

In this case, the subsystem 2 is active. Similar to (4.5), we have

$$
\begin{aligned}
\Delta x_{k+1}(t) &= x_d(t) - x_{k+1}(t) \\
&= \Delta x_{k+1}(t_1) + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t} (t-s)^{\alpha-1} A_2 \Delta x_{k+1}(s) ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t} (t-s)^{\alpha-1} D_2 \Delta x_{k+1}(s-\tau) ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t} (t-s)^{\alpha-1} B_2 \Delta u_{k+1}(s) ds \\
&- \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t} (t-s)^{\alpha-1} A_2 \xi_{k+1}(s) ds,
\end{aligned} \tag{4.24}
$$

When $M_2 = (\Gamma(\alpha) - \|(t)^{\alpha-1} A_2\| - \|(t)^{\alpha-1} D_2\|) > 0$. Using similar manipulation as step 1, we have

$$
\begin{aligned}
\|\Delta x_{k+1}[t_1,t_2]\|_p \le& \frac{\|(t)^{\alpha-1} B_2\|_1 \|\Delta u_{k+1}[t_1,t_2]\|_p + \|(t)^{\alpha-1}\|_1 b_\xi}{M_2} \\
&+ \frac{\Gamma(\alpha)\|\Delta x_{k+1}[t_1,t_2]\|_p}{M_2},
\end{aligned} \tag{4.25}
$$

and

$$
\begin{aligned}
\|\Delta u_{k+1}[t_1,t_2]\|_p &\le (\|I-\Gamma_d C_2 B_2\|_1)\|\Delta u_k[t_1,t_2]\|_p \\
&+ \|\Gamma_p C_2 + \Gamma_d C_2 A_2\|\|\Delta x_k[t_1,t_2]\|_p \\
&+ \|\Gamma_d C_2 D_2\|_p \|\Delta x_k[t_1,t_2-t_M]\|_p + \|\Gamma_d C_2\|_1 b_\xi + \|\Gamma_p\| b_\eta + \|\Gamma_d\| b_{\eta\alpha} \\
&\le (\|I-\Gamma_d C_2 B_2\|_1)\|\Delta u_k[t_1,t_2]\|_p + (\|\Gamma_p C_2 + \Gamma_d C_2 A_2\| + \|\Gamma_d C_2 D_2\|) \\
&\quad \left(\frac{\|(t)^{\alpha-1}B_2\|_1\|\Delta_k[t_1,t_2]\|_p + \|(t)^{\alpha-1}\| b_\xi}{M_2} + \frac{\Gamma(\alpha)\|\Delta x_{k+1}(t_1)\|_p}{M_2}\right) \\
&\qquad + \|\Gamma_d C_2\|_1 b_\xi + \|\Gamma_p\| b_\eta + \|\Gamma_d\| b_{\eta\alpha} \\
&\le \left(\|I-\Gamma_d C_2 B_2\|_1 + \left(\frac{[\|\Gamma_p C_2 + \Gamma_d C_2 A_2\|_1 + \|\Gamma_d C_2 D_2\|]\|(t)^{\alpha-1}B_2\|}{M_2}\right)\right) \\
&\quad \|\Delta u_k[t_1,t_2]\|_p + \left(\frac{[\|\Gamma_p C_2 + \Gamma_d C_2 A_2\|_1 + \|\Gamma_d C_2 D_2\|]\|(t)^{\alpha-1}\|}{M_2}\right) b_\xi \\
&\quad + \|\Gamma_p\| b_\eta + \|\Gamma_d\| b_{\eta\alpha} \\
&\quad + \left(\frac{[\|\Gamma_p C_2 + \Gamma_d C_2 A_2\|_1 + \|\Gamma_d C_2 D_2\|]\Gamma(\alpha)}{M_2}\right)\|\Delta x_{k+1}(t_1)\|_p.
\end{aligned}
$$

So, we have

$$
\begin{aligned}
\|\Delta u_{k+1}[t_1,t_2]\|_p &\le (\|I-\Gamma_d C_2 B_2\| + \beta_2)\|\Delta u_k[t_1,t_2]\|_p + \gamma_2 b_\xi + \|\Gamma_p\| b_\eta \\
&+ \|\Gamma_d\|_{\eta\alpha} + c_2\|\Delta x_k(t_1)\|_p,
\end{aligned} \tag{4.26}
$$

where

$$
\begin{aligned}
\beta_2 &= \left(\|I-\Gamma_d C_2 B_2\|_1 + \left(\frac{[\|\Gamma_p C_2 + \Gamma_d C_2 A_2\|_1 + \|\Gamma_d C_2 D_2\|]\|(t)^{\alpha-1}B_2\|}{M_2}\right)\right), \\
\gamma_2 &= \left(\frac{[\|\Gamma_p C_2 + \Gamma_d C_2 A_2\|_1 + \|\Gamma_d C_2 D_2\|]\|(t)^{\alpha-1}\|}{M_2}\right), \\
c_2 &= \left(\frac{[\|\Gamma_p C_2 + \Gamma_d C_2 A_2\|_1 + \|\Gamma_d C_2 D_2\|]\Gamma(\alpha)}{M_2}\right).
\end{aligned}
$$

Now, from (4.26), we have

$$
\begin{aligned}
\|\Delta u_{k+1}\|_p &\le \rho_2^k\|\Delta u_k[t_1,t_2]\|_p - \rho_2^k\left(\frac{\gamma_2 b_\xi + \|\Gamma_d\| b_\eta + \|\Gamma_d\| b_{\eta\alpha} + c_2\|\Delta x_k(t_1)\|_p}{(1-\rho_2)}\right) \\
&\qquad + \left(\frac{\gamma_2 b_\xi + \|\Gamma_d\| b_\eta + \|\Gamma_d\| b_{\eta\alpha} + c_2\|\Delta x_k(t_1)\|_p}{(1-\rho_2)}\right).
\end{aligned} \tag{4.27}
$$

Now, we have from (4.17)

$$
\lim_{k\to\infty}\|\Delta x_{k+1}(t_1)\|_p \le \|(t)^{\alpha-1}B_1\|\frac{\gamma_1 b_\xi + \|\Gamma_d\| b_\eta + \|\Gamma_d\| b_{\eta\alpha}}{M_1(1-\rho_1)} + \frac{\|(t)^{\alpha-1}\| b_\xi}{M_1}. \tag{4.28}
$$

Computing supremum on both side of (4.27) and taking $k\to\infty$, we have

$$
\begin{aligned}
\lim_{k\to\infty}\sup_{t_1\le t\le t_2}\|\Delta u_{k+1}[t_1,t_2]\|_p &\le \frac{\gamma_2 b_\xi + \|\Gamma_p\| b_\eta + \|\Gamma_d\| b_{\eta\alpha}}{1-\rho_2} \\
&\quad + \frac{c_2\|(t)^{\alpha-1}B_1\|_1(\gamma_1 b_\xi + \|\Gamma_p\| b_\eta + \|\Gamma_d\| b_{\eta\alpha})}{M_1(1-\rho)(1-\rho_2)} \\
&\quad + \left(\frac{c_2\|(t)^{\alpha-1}\|}{M_1(1-\rho_2)}\right).
\end{aligned} \tag{4.29}
$$

Combining (4.25),(4.28) and (4.29), we derive

$$
\begin{aligned}
&\lim_{k\to\infty}\sup_{t_1\le t\le t_2}\|\Delta x_{k+1}[t_1,t_2]\|_p \\
&\quad\le\left[\frac{\gamma_2\|(t)^{\alpha-1}B_2\|_1}{M_2(1-\rho_2)}+\frac{\|(t)^{\alpha-1}\|}{M_2}+\frac{c_2\|(t)^{\alpha-1}B_2\|_1}{M_1M_2(1-\rho_2)}\right.\\
&\quad\left(\frac{\gamma_1\|(t)^{\alpha-1}B_1\|_1}{(1-\rho_1)}+\|(t)^{\alpha-1}\|_1\right)+\\
&\qquad\qquad\left.\frac{\Gamma(\alpha)}{M_1M_2}\left(\frac{\gamma_1\|(t)^{\alpha-1}B_1\|_1}{(1-\rho_1)}+\|(t)^{\alpha-1}\|_1\right)\right]b_\xi,\\
&\quad+\left[\frac{\|(t)^{\alpha-1}B_2\|_1\|\Gamma_p\|}{M_2(1-\rho_2)}+\frac{\|(t)^{\alpha-1}B_1\|\|\Gamma_p\|}{M_1M_2(1-\rho_1)}\left(\frac{c_2\|(t)^{\alpha-1}B_2\|}{1-\rho_2}+\Gamma(\alpha)\right)\right]b_\eta\\
&\quad+\left[\frac{\|(t)^{\alpha-1}B_2\|\|\Gamma_d\|}{M_2(1-\rho_2)}+\frac{\|(t)^{\alpha-1}B_1\|\|\Gamma_d\|}{M_1M_2(1-\rho_1)}\left(\frac{c_2\|(t)^{\alpha-1}B_2\|}{1-\rho_2}+\Gamma(\alpha)\right)\right]b_{\eta\alpha}.
\end{aligned}
$$

So, We have

$$
\lim_{k\to\infty}\sup_{t_1\le t\le t_2}\|\Delta x_{k+1}[t_1,t_2]\|_p\le\omega_2 b_\xi+\varepsilon_2 b_\eta+\lambda_2 b_{\eta\alpha}. \tag{4.30}
$$

where

$$
\begin{aligned}
\omega_2&=\frac{\gamma_2\|(t)^{\alpha-1}B_2\|_1}{M_2(1-\rho_2)}+\frac{\|(t)^{\alpha-1}\|}{M_2}\\
&\quad+\frac{c_2\|(t)^{\alpha-1}B_2\|_1}{M_1M_2(1-\rho_2)}\left(\frac{\gamma_1\|(t)^{\alpha-1}B_1\|_1}{(1-\rho_1)}+\|(t)^{\alpha-1}\|_1\right)\\
&\qquad+\frac{\Gamma(\alpha)}{M_1M_2}\left(\frac{\gamma_1\|(t)^{\alpha-1}B_1\|_1}{(1-\rho_1)}+\|(t)^{\alpha-1}\|_1\right),\\
\varepsilon_2&=\frac{\|(t)^{\alpha-1}B_2\|_1\|\Gamma_p\|}{M_2(1-\rho_2)}+\frac{\|(t)^{\alpha-1}B_1\|\|\Gamma_p\|}{M_1M_2(1-\rho_1)}\left(\frac{c_2\|(t)^{\alpha-1}B_2\|}{1-\rho_2}+\Gamma(\alpha)\right),\\
\lambda_2&=\frac{c_2\|(t)^{\alpha-1}B_2\|\|\Gamma_d\|}{M_2(1-\rho_2)}+\frac{\|(t)^{\alpha-1}B_1\|\|\Gamma_d\|}{M_1M_2(1-\rho_1)}\left(\frac{c_2\|(t)^{\alpha-1}B_2\|}{1-\rho_2}+\Gamma(\alpha)\right).
\end{aligned}
$$

Similar to (4.23), one can conclude that

$$
\lim_{k\to\infty}\sup_{t_1\le t\le t_2}\|\Delta e_{k+1}[t_1,t_2]\|_p\le\omega_2\|C_2\|b_\xi+(\varepsilon_2\|C_2\|b_\eta+1)+\lambda_2\|C_2\|b_{\eta\alpha} \tag{4.31}
$$

Thus, the proof indicates that additionally, uniformly bounded over the interval $[t_1,t_2]$, the tracking error of the subsystem 2.

Step 3: In this proof, we can also get that the tracking errors are also uniformly bounded during the intervals $[t_2,t_3],\dots,[t_{m-1},T]$. □

Therefore, if the conditions of Theorem 1 are given, over the whole range of $[0,T]$, the tracking error is uniformly bounded. That means, throughout the course of the whole interval $[0,T]$, the output of the system asymptotically decreases into a tiny neighbourhood of the intended, desired, or targeted output as the number of iterations increases.

Corollary 4.2. *Assume that the systems (3.4) satisfy Assumptions* $1-5$. *When the systems (3.4) are subjected to the control rule (3.2) without any external noise, if*

(i) $M_i=(\Gamma(\alpha)-\|(t)^{\alpha-1}A_i\|-\|(t)^{\alpha-1}D_i\|)>0,\quad i=1,2,3,\dots,m.$

(ii) $\rho_i = (\|I - \Gamma_d C_i B_i\| + \beta_i) < 1$, *where*

$$\beta_i = \frac{(\|\Gamma_p C_i + \gamma_p C_i A_i\| + \|\Gamma_d C_i D_i\|)\|(t)^{\alpha-1} B_i\|}{M_i},$$

(iii) $\gamma_i = \frac{(\|\Gamma_p C_i + \Gamma_d C_i A_i\|)\|(t)^{\alpha-1}\|}{M_i} + \|\Gamma_d C_i\|, i = 1, 2, 3, ...m.$

Thus, throughout the period $[0, T]$ *as* $k \to \infty$, *the tracking error monotonically converges to zero and the system output converges to the intended output.*

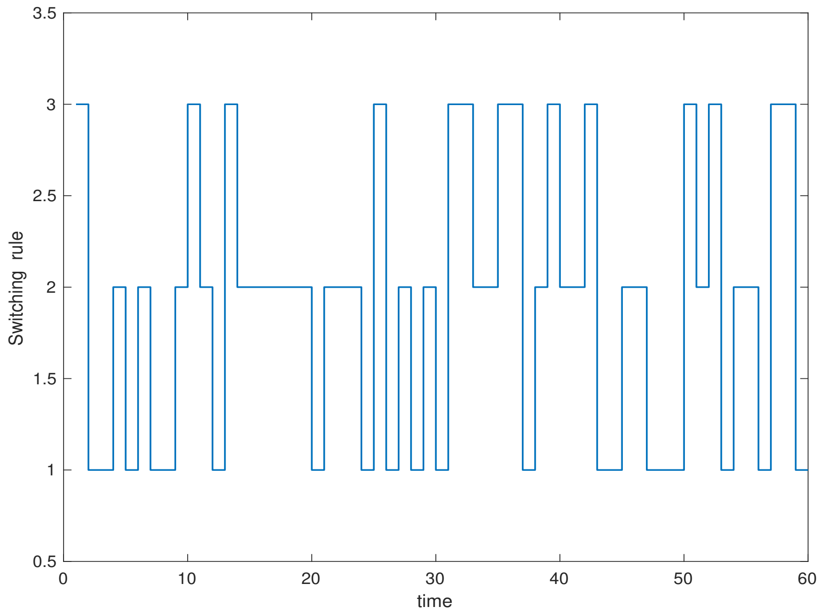

FIGURE 2. Switching rule

5. CONCLUSION

In this paper, we present for a class of fractional-order linear continuous-time delay switched systems with an arbitrary switching sequence the performance of PD$^\alpha$ type fractional order ILC. In certain circumstances, when bounded external noises are present, the PD$^\alpha$-type approach may guarantee that the tracking error is uniformly bound over the whole interval, and when bounded external noises are missing, the tracking error converges monotonically to zero. Further analyse the performance of FOILC algorithms for fractional-order nonlinear switched systems with a time delay input signal.

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