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Research Paper

PSEUDO-SLANT SUBMANIFOLDS OF NEARLY δ- LORENTZIAN TRANS SASAKIAN MANIFOLDS

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ABSRTACT

Our focus is on the existence of certain structures and similarities between pseudo-slant submanifolds and nearly δ-Lorentzian trans Sasakian manifolds. We examine the geometry of these submanifolds. For a totally umbilical proper-slant submanifold that corresponds to a nearly δ- Lorentzian trans Sasakian manifold, we demonstrate necessary and sufficient conditions. Finally, we talk about the integrability of distributions on pseudo-slant submanifolds of a nearly δ-Lorentzian trans-Sasakian manifold.

1. Introduction

While B.Y. Chen [6] defined slant submanifolds as a natural generalization of both holomorphic and fully real immersions, the differential geometry of slant submanifolds has revealed a growing development. These slant submanifolds in nearly Hermitian manifolds have been researched by numerous authors lately. A. Lotta [5] developed the concept of slant submanifolds of a Riemannian manifold into an almost contact metric manifold. Slant submanifolds of Sasakian manifolds were defined by L.J. Cabrerizo et al. in [10]. N. Papaghuic

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[17] introduced and explored the concept of nearly Hermitian manifolds with semi-slant submanifolds. In [3], A. Carriazo defined hemi-slant submanifold. V. A. Khan and colleagues also define the contact version of the pseudo-slant submanifold in a Sasakian manifold in [20]. In [4, 8], authors studied the pseudo-slant submanifold in nearly (ε, δ) trans- Sasakian manifolds and nearly quasi-Sasakian manifolds with connection.

However, Lorentzian para-Sasakian manifolds are a particular kind of almost para-contact metric manifold. The Lorentzian para-Sasakian manifolds concept was first presented by K. Matsumoto [12] in 1989. Subsequently, I. Mihai and R. Rosca [7] separately presented the same idea and produced multiple outcomes on these manifolds. K. Matsumoto, I. Mihai [13], and others have also investigated Lorentzian para-Sasakian manifolds. A more general idea is a nearly δ-Lorentzian trans Sasakian manifold [18]. We explore the nearly δ- Lorentzian trans Sasakian manifold in the current paper. There are significant applications to relativity and conformal mapping, as well as good interaction with other areas of mathematics for nearly δ- Lorentzian trans Sasakian manifolds.

2. Preliminaries

A n-dimensional differentiable manifold $\bar{M}$ has a δ- almost contact metric structure $(f, \xi, \nu, <,>, \delta)$ if it admits a tensor field f of type $(1,1)$, a vector field ξ, a 1-form ν and an indefinite metric $<,>$ satisfying the adherants affinity situations

$$f^2U = U + \nu(U)\xi, \quad \nu(\xi) = -1, \quad <\xi, \xi> = -\delta, \tag{2.1}$$

$$\nu(U) = \delta <U, \xi>, \quad f\xi = 0, \quad \nu(f) = 0, \tag{2.2}$$

$$<fU, fV> = <U, V> + \delta\nu(U)\nu(V), \quad <fU, V> = <U, fV>, \tag{2.3}$$

for all vector fields U,V tangent to M, where $\delta^2 = 1$, so that $\delta = \pm 1$.
A δ-almost contact manifold with sructure $(f, \xi, \nu, <,>, \delta)$ is said to be δ-Lorentzian trans-Sasakian manifold M if it satisfies the conditions

$$(\bar{\nabla}_U f)V = \alpha\{<U, V>\xi - \delta\nu(V)U\} + \beta\{<fU, V> - \delta\nu(V)fU\}, \tag{2.4}$$

for some smooth functions α and β, $\delta = \pm 1$.

Definition 2.1. If $(\bar{\nabla}_U f)V + (\bar{\nabla}_V f)U = \alpha\{2 <U, V>\xi - \delta\nu(V)U - \delta\nu(U)V\} + \beta\{2 <fU, V>\xi - \delta\nu(V)fU - \delta\nu(U)fV\}$, then the manifold on $(\bar{M}, f, \xi, \nu, <,>, \delta)$ is called nearly δ-Lorentzian trans Sasakian manifold.

From Definition 2.1, we have

$$\bar{\nabla}_U \xi = -\delta\alpha fU - \beta\delta f^2U, \tag{2.5}$$

$$(\bar{\nabla}_U \nu)V = \alpha <fU, V> + \beta <fU, fV>. \tag{2.6}$$

Definition 2.2. We say that M is a submanifold of a nearly δ- Lorentzian trans Sasakian manifold $\bar{M}$ if for each non-zero vector X tangent to M at x, the angle $\theta(x) \in [0, \pi/2]$, between ϕX and TX is called the slant angle or the Wirtinger angle of M. If the slant angle is constant for each $X \in \Gamma(TM)$ and $x \in M$, then the submanifold is also called the slant submanifold. If $\theta = 0$, the submanifold is invariant submanifold. If $\theta = \pi/2$ then it is called anti-invariant submanifold. If $\theta(x) \in (0, \pi/2)$, then it is called proper-slant submanifold.

Now, Gauss equation for M in $(\bar{M}, \bar{\nabla})$ is

$$\bar{\nabla}_U V = \nabla_U V + h(U,V), \tag{2.7}$$

and Weingarten formulas are given by

$$\bar{\nabla}_U N = -A_N U + \nabla^{\perp}_U N, \tag{2.8}$$

for $U, V \in TM$ and $N \in T^{\perp}M$. Moreover, we have

$$< A_N U, V >=< h(X,Y), N > . \tag{2.9}$$

For any $U \in TM$ and $N \in T^{\perp}M$, we write

$$fU = TU + NU \quad (TU \in TM \quad and \quad NU \in T^{\perp}M), \tag{2.10}$$

$$\phi N = tN + nN \quad (tN \in TM \quad and \quad nN \in T^{\perp}M). \tag{2.11}$$

Now, we will give the definition of pseudo-slant submanifold which are a generalization of the slant submanifolds.

Definition 2.3. Let M be a pseudo-slant submanifold of nearly δ- Lorentzian trans Sasakian manifold $\bar{M}$, then there exist two orthogonal distributions D_θ and $D^{\perp}$ on M such that
(a) TM admits the orthogonal direct decomposition $TM = D^{\perp} \oplus D_\theta, \quad \xi \in \Gamma(D_\theta)$.
(b) The distribution $D^{\perp}$ is anti-invariant i.e., $\phi(D^{\perp}) \subset T^{\perp}M$.
(c) The distribution D_θ is a slant with slant angle $\theta \neq 0, \pi/2$, that is, the angle between D_θ and $\phi(D_\theta)$ is a constant.

From above, if $\theta = 0$, then the pseudo-slant submanifold is a semi invariant submanifold and if $\theta = \pi/2$, submanifold becomes an anti- invariant.
Suppose M is a pseudo-slant submanifold of nearly δ- Lorentzian trans Sasakian manifold $\bar{M}$ and we denote the dimensions of distributions $D^{\perp}$ and D_θ by c_1 and c_2, respectively, then we have the following cases:
(a) If $c_2 = 0$, then M is an anti-invariant submanifold.
(b) If $c_1 = 0$ and $\theta = 0$, then M is an invariant submanifold.
(c) If $c_1 = 0$ and $\theta \neq 0$, then M is a proper slant submanifold with slant angle θ.
(d) If $c_1.c_2 \neq 0$ and $\theta \in [0, \pi/2]$ then M is a proper pseudo-slant submanifold.

The submanifold M is invariant if N is identically zero. On the other hand, M is anti-invariant if P is identically zero. From (2.1) and (2.10), we have

$$< U, TV >= - < TU, V >, \tag{2.12}$$

for any $U, V \in TM$. If we put $Q = T^2$ we have

$$(\bar{\nabla}_U Q)V = \nabla_U QV - Q\nabla_U V, \tag{2.13}$$

$$(\bar{\nabla}_U T)V = \nabla_U TV - T\nabla_U V, \tag{2.14}$$

$$(\bar{\nabla}_U N)V = \nabla^{\perp}_U NV - N\nabla_U V, \tag{2.15}$$

for any $U, V \in TM$. In view of (2.7) and (2.10), it follows that

$$\bar{\nabla}_U \xi = -\delta\alpha TU - \beta\delta U - \beta\delta\eta(U)\xi, \tag{2.16}$$

$$h(U,\xi) = -\delta\alpha NU. \tag{2.17}$$

The mean curvature vector H of M is given by

$$H = \frac{1}{n}trace(h) = \frac{1}{n}\sum_{i=1}^{n} h(e_i,e_i), \tag{2.18}$$

where n is the dimension of M and $e_1, e_2, ..., e_n$ is a local orthonormal frame of M. A submanifold M of an contact metric manifold $\bar{M}$ is said to be totally umbilical if

$$h(U,V) =< U,V > H, \tag{2.19}$$

there H is a mean curvature vector. A submanifold M is said to be totally geodesic if $h(U,V) = 0$, for each $U,V \in \Gamma(PM)$ and M is said to be minimal if $H = 0$.

3. Pseudo-slant submanifolds of nearly δ-Lorentzian trans Sasakian manifold

The purpose of this section is to study the existence of pseudo-slant submanifolds of nearly δ-Lorentzian trans Sasakian manifold.

Theorem 3.1. *A $(\bar{M}, f, \xi, \nu, <,>, \delta)$ has submanifold of a nearly δ-Lorentzian trans Sasakian manifold such that $\xi \in TM$ then M is a slant if and only if there exists a constant $\lambda \in [0,1]$ such that*

$$T^2 = \lambda\{I + \eta \otimes \xi\}, \tag{3.1}$$

moreover in such a case if θ is the slant angle of M, then $\lambda = cos^2\theta$.

Corollary 3.2. *A $(\bar{M}, f, \xi, \nu, <,>, \delta)$ has slant submanifold of nearly δ-Lorentzian trans Sasakian manifold $\bar{M}$ with slant angle θ, then we have*

$$< TU, TV >= cos^2\theta < fU, fV >, \tag{3.2}$$

$$< NU, NV >= sin^2\theta < fU, fV >, \tag{3.3}$$

where $U,V \in \Gamma(TM)$

From now on, we consider a nearly δ-Lorentzian trans Sasakian manifold $\bar{M}$ and a proper pseudo slant submanifold M. Any vector U tangent to M can be written as

$$U = P_1U + P_2U + \nu(U)\xi, \tag{3.4}$$

where P_1U and P_2U belong to the projections $D^{\perp}$ and D_θ respectively. Now taking f on both sides of equation (3.4), we obtain

$$fX = fP_1U + fP_2U,$$

that is,

$$TU + NU = NP_1U + TP_2U + NP_2U, \tag{3.5}$$

$$TU = TP_2U, \quad NU = NP_1U + NP_2U \quad and \tag{3.6}$$

$$fP_1U = NP_1U, \quad fP_2U = TP_2U + NP_2U, \tag{3.7}$$

$$TP_1U = 0, \quad TP_2U \in \Gamma(D_\theta). \tag{3.8}$$

If we denote the orthogonal complementary of fTM in $D^\perp M$ by μ, then the normal bundle $T^\perp M$ can be decomposed as follows

$$T^\perp M = N(D^\perp) \oplus N(D_\theta) \oplus \mu, \tag{3.9}$$

where μ is an invariant sub bundle of $T^\perp M$ as $N(D^\perp)$ and $N(D_\theta)$ are orthogonal distribution on M. Indeed, $< W, U >= 0$ for each $W \in \Gamma(D^\perp)$ and $U \in \Gamma(D_\theta)$. Thus, by equation (2.1) and (3.8), we can write

$$< NW, NU >=< fW, fU >=< W, U >= 0, \tag{3.10}$$

that is, the distributions $N(D^\perp)$ and $N(D_\theta)$ are mutually perpendicular. In fact, the decomposition (3.9) is an orthogonal direct decomposition.

4. Totally umbilical pseudo-Slant submanifolds of nearly δ-Lorentzian trans Sasakian manifold

We begin with the following Theorem

Theorem 4.1. *A $(\bar{M}, f, \xi, \nu, <, >, \delta)$ has totally umbilical proper pseudo-slant submanifold of nearly δ-Lorentzian trans Sasakian manifold, then M is eithor totally geodesic submanifold or it is an anti-invariant if $H, \nabla^\perp_W H \in \Gamma(\mu)$.*

Proof. Since the ambient space is nearly δ- Lorentzian trans Sasakian manifold, for any $W \in \Gamma(TM)$, by using (2.3), we have

$$\begin{aligned} \bar{\nabla}_W fW - f\bar{\nabla}_W W &= \alpha\{< W, W > \xi - \delta\nu(W)W\} \\ &\quad +\beta\{< fW, W > \xi - \delta\nu(W)fW\}. \end{aligned} \tag{4.1}$$

With the help of equations (2.7), (2.9), (2.10), (2.16) and (4.1), we deduce

$$\begin{aligned} \nabla_W TW &= -< W, TW > H + A_{NW}W - \nabla^\perp_W NW \\ &\quad +f\nabla_W W+ < W, W > fH + \alpha\{< W, W > \xi \\ &\quad -\delta\nu(W)W\} + \beta\{< fW, W > \xi - \delta\nu(W)fW\}. \end{aligned} \tag{4.2}$$

Applying product fH to the above equation we get

$$\begin{aligned} < \nabla^\perp_W NW, fH > &= < N\nabla_W W, fH > + < W, W > \|H\|^2 \\ &\quad +\alpha < W, W >< N\xi, fH > \\ &\quad -\beta\{\delta\nu(W) < NW, fH > \\ &\quad - < NW, W >< N\xi, fH >\}, \end{aligned} \tag{4.3}$$

taking into account (2.8), we get

$$< \bar{\nabla}^\perp_W NW, fH >=< W, W > \|H\|^2. \tag{4.4}$$

Now, for any $W \in \Gamma(TM)$, we obtain

$$(\bar{\nabla}_W f)H = \bar{\nabla}_W fH - f\bar{\nabla}_W H. \tag{4.5}$$

In view of (2.8), (2.10), (2.11), (2.19) and (4.5) we obtain

$$(4.6)\qquad -A_{fH}W = -\nabla^{\perp}_{W}fH + (\bar{\nabla}_{W}f)H - TA_{H}W - NA_{H}W + t\nabla^{\perp}_{W}H + n\nabla^{\perp}_{W}H.$$

Applying product NW to the above equation, we get

$$(4.7)\qquad <\bar{\nabla}_{W}fH, NW> = <(\nabla_{W}n)H + h(tH, W) + NA_{H}W, NW> - <NA_{H}W, NW>.$$

By using (2.9), (2.19) and (3.3), we have

$$<\bar{\nabla}_{W}fH, NW> = sin^{2}\theta\{<W, W> \|H\|^{2} + <h(W, \xi), H> \nu(W)\}.$$

From (2.17), we obtain

$$(4.8)\qquad <\bar{\nabla}_{W}NW, fH> = sin^{2}\theta\{<W, W> \|H\|^{2}\}.$$

Thus, (4.4) and (4.7) imply

$$(4.9)\qquad cos^{2}\theta <W, W> \|H\|^{2} = 0.$$

In such case $H = 0$ and it is minimal. We can state, that eithor M is totally geodesic or it is an anti-invariant submanifold. □

Theorem 4.2. *A $(\bar{M}, f, \xi, \nu, <,>, \delta)$ has totally umbilical pseudo-slant submanifold of nearly δ-Lorentzian trans Sasakian, then at least one of the following statements is true, (i) $dim(D^{\perp}) = 1$, (ii) $H \in \Gamma(\mu)$ and (iii) M is proper pseudo-slant submanifold.*

Proof. Suppose $W \in \Gamma(D^{\perp})$ and the definition of nearly δ-Lorentzian trans Sasakian we conclude that

$$(\bar{\nabla}_{W}f)W = \alpha\{<W, W> \xi - \delta\nu(W)W\} + \beta\{<fW, W> \xi - \delta\nu(W)fW\}.$$

From the last equation, we have

$$(4.10)\qquad -A_{NW}W = th(W, W) + \alpha\{<W, W> P\xi - \nu(W)W\} + \beta\{<TW, W> \xi - \delta\nu(W)TW\}.$$

Taking the product by $X \in \Gamma(D^{\perp})$, we obtain

$$<A_{NW}W + th(W, W) + \alpha\{<W, W> P\xi - \nu(W)W\} + \beta\{<TW, W> P\xi - \delta\nu(W)TW, X>\} = 0$$

It implies that

$$(4.11)\qquad <h(W, X), NW> = -<th(W, W), X> + \alpha\{\delta\nu(W) <W, X> - <W, W><P\xi, X>\} + \beta\{\delta\nu(W) <TW, X> - <TW, W><P\xi, X>\}.$$

Since M is totally umbilical submanifold, we obtain

$$
\begin{aligned}
<W,X><H,NW> \;=\;& -<W,W><tH,X> \\
& -\alpha\{<W,W><P\xi,X> \\
& -\delta\nu(W)<W,X>\} \\
& +\beta\{<P\xi,X><W,TW> \\
& -\delta\nu(W)<W,TX>\}.
\end{aligned}
\tag{4.12}
$$

that is

$$
\begin{aligned}
-<tH,W>X \;=\;& -<tH,X>W-\alpha<P\xi,X>W \\
& +\beta\{<P\xi,X>TW-\delta\nu(W)TZ\} \\
& +\alpha\delta\nu(W)Z.
\end{aligned}
\tag{4.13}
$$

Here tH is either zero or W and Z are linearly dependent vector fields. If $tH \neq 0$, then the vectors W and Z are linearly independent and $dim\Gamma(D^{\perp}) = 1$.
Otherwise $tH = 0$ i.e. $H \in \Gamma(\mu)$. Since $D_\theta \neq 0$, M is pseudo-slant submanifold. Since $\theta \neq 0$ and $c_1.c_2 \neq 0$, M is proper pseudo-slant submanifold. □

Theorem 4.3. *A* $(\bar{M}, f, \xi, \nu, <,>, \delta)$ *has totally umbilical proper pseudo-slant submanifold of nearly* δ*- Lorentzian trans Sasakian manifold, then at least one of the following statements is true;*
(i) $H \in \mu$*,*
(ii) $<\nabla_{TW}\xi, W>= 0$ *,*
(iii) $\nu((\nabla_W T)W>= 0$*,*
(iv) M *is a anti-invariant submanifold.*
(v) If M *proper slant submanifold then,* $dim(M) \geqslant 3$*, for any* $W \in \Gamma(TM)$*.*

Proof. From definition of nearly δ-Lorentzian trans Sasakian manifold and using equations (2.7), (2.8), (2.10) and (2.11), we infer

$$
\begin{aligned}
\nabla_W TW \;=\;& -h(W,TW)+A_{NW}W-\nabla^{\perp}_W NW+T\nabla_W W \\
& +N\nabla_W W+th(W,W)+nh(W,W) \\
& +\beta\{<fW,W>\xi-\delta\nu(W)fW\} \\
& -\alpha\{\delta\nu(W)W-<W,W>\xi\}.
\end{aligned}
\tag{4.14}
$$

Now equating tangential components of last equation, we obtain

$$
\begin{aligned}
\nabla_W TW \;=\;& T\nabla_W W+th(W,W)+A_{NW}W \\
& -\alpha\delta\nu(W)W-\beta\delta\nu(W)TW.
\end{aligned}
\tag{4.15}
$$

By using equations (2.9) and (2.19) and M is a totally umbilical pseudo-slant submanifold, we can write

$$
<A_{NW}W, W>= 0. \tag{4.16}
$$

If $H \in \Gamma(\mu)$, then from (4.14), we obtain

$$
\nabla_W TW = T\nabla_W W+\alpha\delta\nu(W)W-\beta\delta\nu(W)TW.
$$

Taking the product of (4.15) by ξ, we obtain

$$< \nabla_W TW, \xi >= \nu(T\nabla_W W) + \alpha\delta\nu(W)\nu(W) - \beta\delta\nu(W)\nu(TW).$$

$$< \nabla_W TW, \xi >= 0. \tag{4.17}$$

Setting $W = TW$ in last equation, we derive

$$< \nabla_{TW}\xi, T^2W >= 0.$$

In view of equation (3.1), we have

$$cos^2\theta < \nabla_{TW}\xi, (W + \nu(W)\xi) >= 0.$$

Since, M is a proper pseudo slant submanifold, we have

$$< \nabla_{TW}\xi, W >= -\nu(W) < \nabla_{TW}\xi, \xi > . \tag{4.18}$$

For any $W \in \Gamma(TM)$,$< \xi, \xi >= 1$ and taking the covariant derivative of above equation with respect to TW, we have $< \nabla_W\xi, \xi > - < \xi, \nabla_{TW}\xi >= 0$ which implies $< \nabla_{TW}\xi, \xi >= 0$. Hence, with help of equation (4.16), we can state the following results

$$< \nabla_{TW}\xi, W >= 0. \tag{4.19}$$

This proves (ii) of theorem.
Setting W by TW in the equation (4.18), we derive

$$\cos^2\theta < \nabla_W\xi, TW > + \cos^2\theta\nu(W) < \nabla_\xi\xi, TW >= 0,$$

For $\nabla_\xi\xi = 0$, we have

$$\cos^2\theta < \nabla_W\xi, TW >= 0. \tag{4.20}$$

in equation (4.19) if $\cos\theta = 0, \theta = \pi/2$, then M is an anti-invariant submanifold. On the other hand, $< \nabla_w\xi, TW >= 0$, that is $\nabla_W\xi = 0$. This implies that ξ is a Killing vector field in M. If the vector field ξ is not Killing, then we can take at least two linearly independent vectors W and TW to span D_θ, that is, $dim(M) \geqslant 3$. □

5. Integrability of Distributions

In this section we shall discuss the integrability of involved distributions.

Theorem 5.1. *A* $(\bar{M}, f, \xi, \nu, <, >, \delta)$ *has pseudo-slant submanifold of a nearly* δ*- Lorentzian trans Sasakian manifold, then the distribution* $D^\perp \oplus \xi$ *is integrable if and only if for any* $Z, W \in \Gamma(D^\perp \oplus \xi)$

$$\begin{aligned} A_{fW}V &= A_{fV}W + 2[((1 - \alpha\delta)(\nu(W)V - \nu(V)W) \\ &\quad + \beta(1 - \delta)(\nu(TW)V - \nu(TV)W)]. \end{aligned}$$

Proof. For any $W, V \in \Gamma(D^\perp \oplus \xi)$ and $U \in \Gamma(TM)$, then from (2.9), we have

$$2 < A_{fW}V, U >=< h(U, V), fW > + < h(U, V), fW),$$

In view of (2.7), the above equation reduce to

$$\begin{aligned} 2 < A_{fW}V, U > &= < \bar{\nabla}_V U, fW) + < \bar{\nabla}_U V, fW > \\ &=< f\bar{\nabla}_V U, W > + < f\bar{\nabla}_U V, W > . \end{aligned}$$

So we have

$$\begin{aligned} 2 < A_{fW}V, U) &= < \bar{\nabla}_V fU, W > + < \bar{\nabla}_U fV, W > \\ &\quad - < (\bar{\nabla}_V f)U + < (\bar{\nabla}_U f)V, W > . \end{aligned}$$

By the definition of nearly δ-Lorentzian trans Sasakian we conclude that

$$\begin{aligned} 2 < A_{\phi W}V, U > &= < T\nabla_V W + th(W,V) - A_{fV}W, U > \\ &\quad -2 << V, \xi > W, U > +\alpha\delta << V, \xi > W, U > \\ &\quad -2\beta << TV, \xi > W, U > -2\beta << NV, \xi > W, V > \\ &\quad +\beta\delta << TV, \xi > W, U > +\beta\delta << TV, \xi > W, U > \\ &\quad +\beta\delta << TV, \xi > W, U > +\beta\delta << NV, \xi > W, U > \\ &\quad +\alpha\delta << V, \xi > W, U > . \end{aligned}$$

which is equivalent to

$$\begin{aligned} 2A_{fW}V &= T\nabla_V W + th(W,V) - A_{fV}W - 2\nu(V)W \\ &\quad +2\alpha\delta\nu(V)W - 2\beta\{\nu(TV)W - \delta\nu(TV)W\}. \end{aligned} \tag{5.1}$$

Take $W = V$ in (5.1), we infer

$$\begin{aligned} 2A_{\phi V}W &= T\nabla_W V + th(V,W) - A_{fW}V - 2\nu(W)V \\ &\quad +2\alpha\delta\nu(W)V - 2\beta\{\nu(TW)V - \delta\nu(TV)W\} \end{aligned} \tag{5.2}$$

By using equations (5.1) and (5.2), we conclude that

$$\begin{aligned} A_{fW}V - A_{fV}W &= 2[(1-\alpha\delta)\{\nu(W)V - \nu(V)W\} \\ &\quad +\beta(1-\delta)\{\nu(TW)V - \nu(TV)W\}]. \end{aligned} \tag{5.3}$$

Therefore, the distribution $D^{\perp} \oplus \xi$ is integrable if and only if $T[W,V] = 0$ which proves our assertion. □

Theorem 5.2. *A $(\bar{M}, f, \xi, \nu, <,>, \delta)$ has pseudo-slant submanifold of a nearly δ- Lorentzian trans Sasakian manifold, then the slant distribution D_θ is integrable if and only if for any $X, Y \in \Gamma(D_\theta)$*

$$\begin{aligned} & P_1\{\nabla_U TV - T\nabla_V U + (\nabla_V T)U - A_{NU}V - A_{NV}U \\ & -2th(U,V) + \alpha\delta\nu(V)U + \alpha\delta\nu(U)V + \beta\delta\nu(V)TU \\ & +\beta\delta\nu(U)TV\} = 0. \end{aligned} \tag{5.4}$$

Proof. For all $U, V \in \Gamma(D_\theta)$ and we denote the projections on $D^{\perp}$ and D_θ by P_1 and P_2 , respectively, then for any vector fields $U, V \in \Gamma(D_\theta)$, we obtain

$$\begin{aligned} \bar{\nabla}_U fV &= f\bar{\nabla}_U V - \bar{\nabla}_V fU + f\bar{\nabla}_V U \\ &\quad +2\alpha\{< U, V > \xi - \delta\nu(V)U - \delta\nu(U)V\} \\ &\quad +\beta\{< fU, V > \xi - \delta\nu(V)fU - \delta\nu(U)fV\}. \end{aligned}$$

With the help of equations (2.7), (2.8), (2.10) and (2.11), we find

$$\begin{aligned}\bar{\nabla}_U TV &= -\bar{\nabla}_U NV + f(\nabla_U V + h(U,V)) - \bar{\nabla}_V TU - \bar{\nabla}_V NU \\ &+\alpha\{2 < U,V > \xi - \delta\nu(V)U - \delta\nu(U)V\} \\ &+\beta\{2 < fU,V > \xi - \delta\nu(V)fU - \delta\nu(U)fV\} \\ &+f(\nabla_V U + h(U,V)).\end{aligned}$$

$$\begin{aligned}\nabla_T V &= -h(U,TV) + A_{NV}U - \nabla^{\perp}_U NV + T\nabla_U V \\ &+N\nabla_U V + th(U,V) + nh(U,V) - \nabla_V TU \\ &-\nabla^{\perp}_V NU + T\nabla_V U + N\nabla_V U + th(U,V) \\ &+\alpha\{2 < U,V > \xi - \delta\nu(V)U - \delta\nu(U)V\} \\ &+\beta\{2 < fU,V > \xi - \delta\nu(V)fU - \delta\nu(U)fV\} \\ &-h(V,TU) + nh(U,V) + A_{NU}V.\end{aligned} \tag{5.5}$$

Now comparing tangential parts of the last equation, we have

$$\begin{aligned}\nabla_U TV &= T\nabla_U V - (\nabla_V T)U + A_N UV + A_N VU \\ &+\alpha\{2 < U,V > \xi - \delta\nu(V)U - \delta\nu(U)V\} \\ &+\beta\{2 < fU,V > \xi - \delta\nu(V)fU - \delta\nu(U)fV\} \\ &+2th(U,V).\end{aligned} \tag{5.6}$$

$$\begin{aligned}T[U,V] &= \nabla_U TV - T\nabla_U V + (\nabla_V T)U - A_{NU}V \\ &-A_{NV}U - 2th(U,V) + \alpha\delta\{\nu(V)U + \nu(U)V\} \\ &+\beta\delta\{\nu(V)TU + \nu(U)TV\}.\end{aligned} \tag{5.7}$$

Applying P_1 to (5.7), we get (5.4) □

Theorem 5.3. *In a pseudo-slant submanifold of nearly δ- Lorentzian trans Sasakian manifold is given by*

$$\begin{aligned}(\nabla_U T)V &= A_{NV}U + A_{NU}V + th(U,V) + T(h(U,V)) \\ &+\alpha\{2 < U,V > \xi - \delta\nu(V)U - \delta\nu(U)V\} \\ &+\beta\{2 < TU,V > \xi - \delta\nu(V)TU - \delta\nu(U)TV\} \\ &-(\nabla_V T)U.\end{aligned} \tag{5.8}$$

Proof. $\forall\ U,V \in TM$, using equations (2.9), (2.10) and (2.11), we infer

$$\begin{aligned}\bar{\nabla}_U TV &= -\bar{\nabla}_U NV + (\bar{\nabla}_U f)V + T(\nabla_U V) \\ &+N(\nabla_X UV) + th(U,V) + nh(U,V)\end{aligned}$$

Using (2.8) and (2.9) from above, we obtain

$$\begin{aligned}\nabla_U TV &= -h(U,TV) + A_{NV}U - \nabla^{\perp}_U NV \\ &+(\bar{\nabla}_U f)V + (\bar{\nabla}_V f)U + th(U,V) + nh(U,V) \\ &-(\bar{\nabla}_V f)U + T(\nabla_U V) + N(\nabla_U V),\end{aligned} \tag{5.9}$$

we obtained,

$$\begin{aligned}2\alpha < U,V > \xi &= \alpha\delta\{\nu(V)U + \nu(U)V\} + \bar{\nabla}_V fU - f(\bar{\nabla}_U)\\ &+\beta\delta\{\nu(V)fU + \nu(U)fV\} - 2\beta < fU,V > \xi\\ &-T\nabla_U V - N\nabla_U V - th(U,V) - nh(U,V)\\ &+\nabla_U TV + h(U,TV) - A_{NV}U + \nabla^{\perp}_U NV.\end{aligned}$$

Now equating tangential and normal component parts of the last equation, we have

$$\begin{aligned}\nabla_U TV &= A_{NV}U + 2\alpha < U,V > \xi - \alpha\delta\nu(V)U\\ &+2\beta < NU,V > \xi - \beta\delta\nu(V)TV - \beta\delta\nu(V)NU\\ &+2\beta < TU,V > \xi - \beta\delta\nu(U)NV - \bar{\nabla}_V TU\\ &+N(\bar{\nabla}_V U) + T\nabla_U V + N\nabla_U V + th(U,V)\\ &-\alpha\delta\nu(U)V - \beta\delta\nu(U)TV - \bar{\nabla}_V NU\\ &+T(\bar{\nabla}_V U) + nh(U,V),\end{aligned} \tag{5.10}$$

That is,

$$\begin{aligned}(\nabla_U T)V &= A_{NV}U + A_{NU}V + th(U,V) + T(h(V,U)\\ &+\alpha\{2 < U,V > \xi - \delta\nu(V)U - \delta\nu(U)V\\ &+\beta\{2 < TU,V > \xi - \delta\nu(V)TU - \delta\nu(U)TV\}\\ &-(\nabla_V T)U.\end{aligned} \tag{5.11}$$

□

Theorem 5.4. *A $(\bar{M}, f, \xi, \nu, <,>, \delta)$ has pseudo-slant submanifold of nearly δ- Lorentzian trans Sasakian manifold, then we have*

$$\begin{aligned}A_{fV}U &= -A_{fU}V + \delta\{\alpha < U,V > \xi + \beta < fU,V >\}\xi\\ &+\nabla_U(TV) + h(U,TV) - A_{NV}U + \nabla^{\perp}_U NV\\ &-T(\nabla_U V) - N(\nabla_U V) - Nh(U,V),\end{aligned} \tag{5.12}$$

$\forall\ U,V \in D^{\perp}$.

Proof. In view of equation (2.8), we have

$$< A_{fV}U,W >=< h(U,W), fV >=< fh(U,W),V > . \tag{5.13}$$

From equations (2.7), (5.13) and $\phi\nabla_W U \in T^{\perp}M$, we conclude that

$$< A_{fV}U,W >=< \bar{\nabla}_W fU,V > - < (\bar{\nabla}_W f)U,V > . \tag{5.14}$$

Now, for $U \in D_1$, $\quad \phi X \in T^{\perp}M$. Hence, from (2.8) we have

$$\bar{\nabla}_W fU = -A_{fU}W + \nabla^{\perp}_W fU. \tag{5.15}$$

With the help of equations (5.14) and (5.15), we find

$$< A_{fV}U,W) = - < (\bar{\nabla}_W f)U,V > - < A_{fU}W,V > . \tag{5.16}$$

Since $h(U,V)=h(V,U)$, if follows from (2.9) that

$$< A_{fU}W,V>=< A_{fU}V,W> .$$

Hence, from (5.16) we obtain, with the help of (2.4),

$$\begin{aligned} < A_{fV}U,W> &= -< A_{fU}V,W> -\nu(W)<U,V> \\ &\quad -2\nu(V)<U,W> -4\nu(U)\nu(V)\nu(W) \\ &\quad +<\nabla_U(TV)+h(U,TV)-A_{NV}U \\ &\quad +\nabla^{\perp}_U NV - T(\nabla_U V)-N(\nabla_U V) \\ &\quad -Th(U,V)-Nh(U,V),W> \\ &\quad -\nu(U)<W,V> . \end{aligned} \tag{5.17}$$

Since $U,V,W\in D^{\perp}$ and an orthogonal distribution to the ξ, it follows that $\eta(U)=\eta(V)=0$, the above equations reduces to,

$$\begin{aligned} A_{fV}U &= -A_{fU}V+\delta\{\alpha<U,V>\xi+\beta<fU,V>\xi\} \\ &\quad +\nabla_U(TV)+h(U,TV)-A_{NV}U+\nabla^{\perp}_U NV \\ &\quad -T(\nabla_U V)-N(\nabla_U V)-Nh(U,V). \end{aligned}$$

□

Theorem 5.5. *A* $(\bar{M},f,\xi,\nu,<,>,\delta)$ *has pseudo-slant of nearly* δ*- Lorentzian trans Sasakian manifold, then the anti-invariant distribution* $D^{\perp}$ *is integrable if and only if for any* $Z,W\in\Gamma(D^{\perp})$

$$\begin{aligned} A_{NW}U &= -A_{NU}W-2T\nabla_W U+\alpha\delta\{\nu(U)W+\nu(W)U\} \\ &\quad -2th(W,U)+\beta\delta\{\nu(W)TU+\nu(U)TW\} \\ &\quad -2\{\alpha<U,W>\xi+\beta<TU,W>\xi\}. \end{aligned} \tag{5.18}$$

Proof. For any $U,W\in\Gamma(D^{\perp})$ and the definition of nearly δ-Lorentzian trans Sasakian we conclude that

$$\begin{aligned} \bar{\nabla}_U fW &= f\bar{\nabla}_U W-\bar{\nabla}_W fU+f\bar{\nabla}_W U \\ &\quad +\alpha(2<U,W>\xi-\delta\nu(W)U-\delta\nu(U)W) \\ &\quad +\beta(2<fU,W>\xi-\delta\nu(W)fU-\delta\nu(U)fW). \end{aligned}$$

With the help of equations (2.7), (2.8), (2.10) and (2.11), we deduce

$$\begin{aligned} 2\alpha<U,W>\xi &= \alpha\delta\{\nu(W)U+\nu(U)W\}-2\beta<fU,W>\xi \\ &\quad +\beta\delta\{\nu(W)fU+\nu(U)fW\}-A_{NW}U+\nabla^{\perp}_U NW \\ &\quad -T\nabla_U W-N\nabla_U W-2th(W,U)-A_{NU}W \\ &\quad +\nabla^{\perp}_W NU-T\nabla_W U-N\nabla_W U-2nh(W,U) \end{aligned}$$

Now equating the tangential parts of last equation, we have

$$\begin{aligned} -2\alpha<U,W>\xi &= -\alpha\delta\nu(W)U+2\beta<TU,W>\xi+A_{NW}U \\ &\quad -\alpha\delta\nu(U)W-\beta\delta\eta(W)TU-\beta\delta\nu(U)TW \\ &\quad +T\nabla_U W+2th(W,U)+A_{NU}W+T\nabla_W U \end{aligned}$$

From the above equation, we can infer

$$
\begin{aligned}
T[U,W] &= -A_{NW}U - A_{NU}W - 2T\nabla_W U - 2th(W,U) \\
&\quad -2\alpha < U,W > \xi + \alpha\delta\{\nu(W)U + \nu(U)W\} \\
&\quad -2\beta < TU,W > \xi + \beta\delta\{\nu(W)TU + \nu(U)TW\}
\end{aligned}
$$

Thus $[Z,W] \in \Gamma(D^{\perp})$ if and only if (5.11) is satisfied. $\square$

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