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Research Paper

# ON GENERALIZED BERWALD $(\alpha, \beta)$-MANIFOLDS WITH RELATIVELY ISOTROPIC LANDSBERG CURVATURE

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ABSTRACT

The class of generalized Berwald metrics contains the class of Berwald metrics as a special case. Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be a generalized Berwald $(\alpha, \beta)$-metric on manifold $M$. We show that $F$ has vanishing S-curvature $\mathbf{S} = 0$ and is of relatively isotropic Landsberg curvature $\mathbf{L} + cF\mathbf{C} = 0$ if and only if $\mathbf{B} = 0$, where $c = c(x)$ is a scalar function on $M$.

## 1. Introduction

A Finsler metric $F$ on a $C^\infty$ manifold $M$ is called a generalized Berwald metric if there exists a covariant derivative $\nabla$ on $M$ such that the parallel translations induced by $\nabla$ preserve the Finsler function $F$ [11][14]. In this case, $(M, F)$ is called a generalized Berwald manifold. If $\nabla$ is also torsion-free, then $F$ reduces to a Berwald metric. Also, one can define a Berwald metric during the spray coefficients. Let $(M, F)$ be a Finsler manifold. The Finsler metric $F$ on $M$ induced a spray

$$\mathbf{G} = y^i \frac{\partial}{\partial x^i} - 2G^i(x, y)\frac{\partial}{\partial y^i}$$

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which determines the geodesics, where $G^i = G^i(x, y)$ are called the spray coefficients of $\mathbf{G}$. A Finsler metric $F$ is called a Berwald metric if $G^i = \frac{1}{2}\Gamma^i_{jk}(x)y^j y^k$ are quadratic in $y \in T_xM$ for any $x \in M$. The Berwald curvature $\mathbf{B}$ of Finsler metrics is an important non-Riemannian quantity constructed by L. Berwald. Then, every Berwald metric is a trivially generalized Berwald metric. The main interesting point about the class of generalized Berwald manifolds lies in the fact that these manifolds may have a rich isometry group [9][10]. For the recent progress about the class of generalized Berwald manifolds, see [11], [16] and [14].

Beside the Berwald curvature, there is another interesting non-Riemannian quantity that is close to the Berwald curvature, namely, S-curvature. The S-curvature $\mathbf{S}$ is constructed by Shen for given comparison theorems on Finsler manifolds [8]. An interesting problem in Finsler geometry is to study and characterize Finsler metrics of vanishing S-curvature. It is known that some of Randers metrics are of vanishing S-curvature [7][13]. This is one of our motivations to consider Finsler metrics with vanishing S-curvature. Shen proved that every Berwald metric satisfies $\mathbf{S} = 0$ [8].

There are two basic tensors on Finsler manifolds: fundamental metric tensor $\mathbf{g}_y$ and the Cartan torsion $\mathbf{C}_y$, which are second and third order derivatives of $\frac{1}{2}F_x^2$ at $y \in T_xM_0$, respectively. It is easy to see that every Finsler metric with vanishing Cartan torsion is a Riemannian metric. The rate of change of $\mathbf{C}$ along Finslerian geodesics is called Landsberg curvature $\mathbf{L}_y$. A Finsler metric with vanishing Landsberg curvature is called a Landsberg metric. In [15], Vincze et al. studied generalized Berwald surface with vanishing Landsberg curvature and proved the following.

**Theorem 1.1.** ([15]) Every connected generalized Berwald surface is a Landsberg surface if and only if it is a Berwald surface.

It is obvious that $\mathbf{L}/\mathbf{C}$ can be regarded as the relative rate of change of Cartan torsion $\mathbf{C}$ along Finslerian geodesics. Then $F$ is said to be relatively isotropic Landsberg metric if $\mathbf{L} + cF\mathbf{C} = 0$, where $c = c(x)$ is a scalar function on $M$. If $c = 0$, then $F$ reduces to a Landsberg metric. In order to find some Finsler metrics of relatively isotropic Landsberg curvature, one can consider the class of $(\alpha, \beta)$-metrics. An $(\alpha, \beta)$-metric is a Finsler metric on $M$ defined by $F := \alpha\phi(s)$, where $s = \beta/\alpha$, $\phi = \phi(s)$ is a $C^\infty$ function on the $(-b_0, b_0)$ with certain regularity, $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a positive-definite Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form on $M$. The simplest $(\alpha, \beta)$-metrics are the Randers metrics $F = \alpha + \beta$ which were discovered by G. Randers when he studied 4-dimensional general relativity. In [14], Vincze proved that a Randers metric $F = \alpha + \beta$ is a generalized Berwald metric if and only if dual vector field $\beta^\sharp$ is of constant Riemannian length. In [11], Tayebi-Barzegari showed that an $(\alpha, \beta)$-metric satisfying the so-called sign property is a generalized Berwald metric if and only if $\beta^\sharp$ is of constant Riemannian length. Then, Vincze showed that an $(\alpha, \beta)$-metric satisfying $\phi'(0) \neq 0$ is a generalized Berwald metric if and only if $\beta^\sharp$ is of constant Riemannian length [16]. In this paper, we study the class of generalized Berwald $(\alpha, \beta)$-metrics with relatively isotropic Landsberg curvature and vanishing S-curvature. We find that such metrics must be Berwaldian. More precisely, we prove the following.

**Theorem 1.2.** *Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be a generalized Berwald $(\alpha, \beta)$-metric on manifold $M$ such that $\phi'(0) \neq 0$. Then $F$ has vanishing S-curvature $\mathbf{S} = 0$ and is of relatively isotropic*

*Landsberg curvature, namely* $\mathbf{L}/\mathbf{C}$ *is isotropic,*

$$\mathbf{L} + c(x)F\mathbf{C} = 0, \tag{1.1}$$

*where* $c = c(x)$ *is a scalar function on* $M$ *if and only if* $\mathbf{B} = 0$*.*

Theorem 1.2 can be considered as a local extension of Theorem 1.1. Also, by using Theorem 1.2, one can conclude the following.

**Corollary 1.3.** *Let* $F = \alpha\phi(s)$*,* $s = \beta/\alpha$*, be a non-Randers type generalized Berwald* $(\alpha, \beta)$*-metric on manifold* $M$ *of dimension* $n \geq 3$ *such that* $\phi'(0) \neq 0$*. Then* $F$ *has vanishing E-curvature* $\mathbf{E} = 0$ *and is of relatively isotropic Landsberg curvature* $\mathbf{L} + c(x)F\mathbf{C} = 0$ *if and only if* $\mathbf{B} = 0$*, where* $c = c(x)$ *is a scalar function on* $M$*.*

In this paper, we use the Berwald connection and the $h$- and $v$- covariant derivatives of a Finsler tensor field are denoted by " | " and ", " respectively [5].

## 2. Preliminary

A Finsler metric on a manifold $M$ is a nonnegative function $F$ on $TM$ having the following properties

(a) $F$ is $C^\infty$ on $TM_0 := TM \setminus \{0\}$;
(b) $F(\lambda y) = \lambda F(y)$, $\forall \lambda > 0$, $y \in TM$;
(c) for each $y \in T_xM$, the following quadratic form $\mathbf{g}_y$ on $T_xM$ is positive definite,

$$\mathbf{g}_y(u, v) := \frac{1}{2}\Big[F^2(y + su + tv)\Big]\Big|_{s,t=0}, \qquad u, v \in T_xM.$$

Then the pair $(M, F)$ is called a Finsler manifold.

At each point $x \in M$, $F_x := F|_{T_xM}$ is an Euclidean norm if and only if $\mathbf{g}_y$ is independent of $y \in T_xM_0$. To measure the non-Euclidean feature of $F_x$, define $\mathbf{C}_y : T_xM \times T_xM \times T_xM \to \mathbb{R}$ by

$$\mathbf{C}_y(u, v, w) := \frac{1}{2}\frac{d}{dt}\Big[\mathbf{g}_{y+tw}(u, v)\Big]\Big|_{t=0}, \qquad u, v, w \in T_xM.$$

The family $\mathbf{C} := \{\mathbf{C}_y\}_{y \in TM_0}$ is called the Cartan torsion.

Given a Finsler manifold $(M, F)$, then a global vector field $\mathbf{G}$ is induced by $F$ on $TM_0$, which in a standard coordinate $(x^i, y^i)$ for $TM_0$ is given by

$$\mathbf{G} = y^i\frac{\partial}{\partial x^i} - 2G^i(x, y)\frac{\partial}{\partial y^i},$$

where $G^i(x, y)$ are local functions on $TM_0$ satisfying $G^i(x, \lambda y) = \lambda^2 G^i(x, y)$, $\lambda > 0$. $\mathbf{G}$ is called the associated spray to $(M, F)$. The projection of an integral curve of $G$ is called a geodesic in $M$. In local coordinates, a curve $c(t)$ is a geodesic if and only if its coordinates $(c^i(t))$ satisfy

$$\ddot{c}^i + 2G^i(\dot{c}) = 0.$$

Using the spray of $F$, one can define $\mathbf{B}_y : T_xM \times T_xM \times T_xM \to T_xM$ by $\mathbf{B}_y(u, v, w) := B^i{}_{jkl}(y)u^jv^kw^l\partial/\partial x^i|_x$, where

$$B^i{}_{jkl} := \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l}.$$

$\mathbf{B}$ is called the Berwald curvature.

Define the mean of Berwald curvature by $\mathbf{E}_y : T_xM \times T_xM \to \mathbb{R}$, where

$$\mathbf{E}_y(u,v) := \frac{1}{2}\sum_{i=1}^{n} g^{ij}(y)\mathbf{g}_y\Big(\mathbf{B}_y(u,v,\partial_i),\partial_j\Big).$$

The family $\mathbf{E} = \{\mathbf{E}_y\}_{y\in TM_0}$ is called the mean Berwald curvature or E-curvature of $F$. In a local coordinates, $\mathbf{E}_y(u,v) := E_{ij}(y)u^iv^j$, where

$$E_{ij} := \frac{1}{2}B^m{}_{mij}.$$

A Finsler metric $F$ is called a weakly Berwald metric if $\mathbf{E} = 0$.

Let $U(t)$ be a vector field along a curve $c(t)$. The canonical covariant derivative $\mathrm{D}_{\dot c}U(t)$ is defined by

$$\mathrm{D}_{\dot c}U(t) := \Big\{\frac{dU^i}{dt}(t) + U^j(t)\frac{\partial G^i}{\partial y^j}(\dot c(t))\Big\}\frac{\partial}{\partial x^i}|_{c(t)}.$$

$U(t)$ is said to be parallel along $c$ if $\mathrm{D}_{\dot c(t)}U(t) = 0$.

To measure the changes of the Cartan torsion $\mathbf{C}$ along geodesics, we define $\mathbf{L}_y : T_xM \times T_xM \times T_xM \to \mathbb{R}$ by

$$\mathbf{L}_y(u,v,w) := \frac{d}{dt}\Big[\mathbf{C}_{\dot c(t)}(U(t),V(t),W(t))\Big]\Big|_{t=0},$$

where $c(t)$ is a geodesic and $U(t), V(t), W(t)$ are parallel vector fields along $c(t)$ with $\dot c(0) = y, U(0) = u, V(0) = v, W(0) = w$. The family $\mathbf{L} := \{\mathbf{L}_y\}_{y\in TM\setminus\{0\}}$ is called the Landsberg curvature. A Finsler metric is called a Landsberg metric if $\mathbf{L} = 0$. An important fact is that if $F$ is Berwaldian, then it is Landsbergian. $\mathbf{L}/\mathbf{C}$ is regarded as the relative rate of change of $\mathbf{C}$ along Finslerian geodesics. Then $F$ is said to be isotropic Landsberg metric if $\mathbf{L} = cF\mathbf{C}$, where $c = c(x)$ is a scalar function on $M$.

For a Finsler metric $F$ on an $n$-dimensional manifold $M$, the Busemann-Hausdorff volume form $dV_F = \sigma_F(x)dx^1\cdots dx^n$ is defined by

$$\sigma_F(x) := \frac{\mathrm{Vol}B^n(1)}{\mathrm{Vol}\Big\{(y^i)\in\mathbb{R}^n \ \Big|\ F\Big(y^i\frac{\partial}{\partial x^i}|_x\Big) < 1\Big\}}.$$

In general, the local scalar function $\sigma_F(x)$ can not be expressed in terms of elementary functions, even $F$ is locally expressed by elementary functions.

Let $G^i(x,y)$ denote the geodesic coefficients of $F$ in the same local coordinate system. The S-curvature is defined by

$$\mathbf{S}(\mathbf{y}) := \frac{\partial G^i}{\partial y^i}(x,y) - y^i\frac{\partial}{\partial x^i}\Big[\ln\sigma_F(x)\Big].$$

where $\mathbf{y} = y^i\frac{\partial}{\partial x^i}|_x \in T_xM$. It is proved that $\mathbf{S} = 0$ if $F$ is a Berwald metric [7]. There are many non-Berwald metrics satisfying $\mathbf{S} = 0$ [1].

Given a Riemannian metric $\alpha$, a 1-form $\beta$ on a manifold $M$, and a $C^\infty$ function $\phi = \phi(s)$ on $[-b_o, b_o]$, where $b_o := \sup_{x\in M} \|\beta\|_x$, one can define a function on $TM$ by

$$F := \alpha\phi(s), \qquad s = \frac{\beta}{\alpha}.$$

If $\phi$ and $b_o$ satisfy (2.1) and (2.2) below, then $F$ is a Finsler metric on $M$. Finsler metrics in this form are called $(\alpha, \beta)$-metrics. Randers metrics are special $(\alpha, \beta)$-metrics.

Now we consider $(\alpha, \beta)$-metrics. Let $\alpha = \sqrt{a_{ij}y^iy^j}$ be a Riemannian metric and $\beta = b_iy^i$ a 1-form on a manifod $M$. Let

$$\|\beta\|_x := \sqrt{a^{ij}(x)b_i(x)b_j(x)}.$$

For a $C^\infty$ function $\phi = \phi(s)$ on $[-b_o, b_o]$, where $b_o = \sup_{x\in M} \|\beta\|_x$, define

$$F := \alpha\phi(s), \quad s = \frac{\beta}{\alpha}.$$

By a direct computation, we obtain

$$g_{ij} = \rho a_{ij} + \rho_0 b_ib_j - \rho_1(b_i\alpha_j + b_j\alpha_i) + s\rho_1\alpha_i\alpha_j,$$

where $\alpha_i := a_{ij}y^j/\alpha$, and

$$\begin{aligned}
\rho &:= \phi(\phi - s\phi'),\\
\rho_0 &:= \phi\phi'' + \phi'\phi',\\
\rho_1 &:= s(\phi\phi'' + \phi'\phi') - \phi\phi'.
\end{aligned}$$

By further computation, one obtains

$$\det(g_{ij}) = \phi^{n+1}\left(\phi - s\phi'\right)^{n-2}\Big[(\phi - s\phi') + (\|\beta\|_x^2 - s^2)\phi''\Big]\det(a_{ij}).$$

Using the continuity, one can easily show that

**Lemma 2.1.** *Let $b_o > 0$. $F = \alpha\phi(\beta/\alpha)$ is a Finsler metric on $M$ for any pair $\{\alpha, \beta\}$ with $\sup_{x\in M}\|\beta\|_x \le b_o$ if and only if $\phi = \phi(s)$ satisfies the following conditions:*

$$\phi(s) > 0, \qquad (|s| \le b_o) \tag{2.1}$$

$$\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0, \qquad (|s| \le b \le b_o). \tag{2.2}$$

Let

$$r_{ij} := \frac{1}{2}(b_{i;j} + b_{j;i}), \quad s_{ij} := \frac{1}{2}(b_{i;j} - b_{j;i}), \quad r_{i0} := r_{ij}y^j, \quad r_{00} := r_{ij}y^iy^j, \quad r_j := b^ir_{ij},$$
$$s_{i0} := s_{ij}y^j, \quad s_j := b^is_{ij}, \quad s^i{}_j = a^{im}s_{mj}, \quad s^i{}_0 = s^i{}_jy^j, \quad r_0 := r_jy^j, \quad s_0 := s_jy^j.$$

Suppose that $G^i = G^i(x, y)$ and $\bar{G}^i = \bar{G}^i(x, y)$ denote the coefficients of $F$ and $\alpha$ respectively in the same coordinate system. By definition, we obtain the following identity

$$G^i = \bar{G}^i + Py^i + Q^i, \tag{2.3}$$

where

$$
\begin{aligned}
P &= \alpha^{-1}\Theta\Big[r_{00} - 2Q\alpha s_0\Big] \\
Q^i &= \alpha Q s^i{}_0 + \Psi\Big[r_{00} - 2Q\alpha s_0\Big] b^i, \\
Q &= \frac{\phi'}{\phi - s\phi'} \\
\Theta &= \frac{\phi\phi' - s(\phi\phi'' + \phi'\phi')}{2\phi\Big((\phi - s\phi') + (b^2 - s^2)\phi''\Big)} \\
\Psi &= \frac{1}{2}\frac{\phi''}{(\phi - s\phi') + (b^2 - s^2)\phi''}.
\end{aligned}
$$

Clearly, if $\beta$ is parallel with respect to $\alpha$ ($r_{ij} = 0$ and $s_{ij} = 0$), then $P = 0$ and $Q^i = 0$. In this case, $G^i = \bar{G}^i$ are quadratic in $y$, and $F$ is a Berwald metric.

## 3. Proof of Theorem 1.2

In this section, we will prove a generalized version of Theorem 1.2. Indeed, we study generalized Berwald $(\alpha, \beta)$-metric with relatively isotropic mean Landsberg curvature and isotropic S-curvature. More precisely, we prove the following.

**Theorem 3.1.** *Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be an non-Riemannian generalized Berwald $(\alpha, \beta)$-metric on manifold $M$ such that $\phi \neq c_1\sqrt{1 + c_2 s^2} + c_3 s$ and $\phi'(0) \neq 0$ for any constant $c_1 > 0$, $c_2$, $c_3$. Then $F$ has isotropic S-curvature $\mathbf{S} = (n+1)\lambda F$ and is of relatively isotropic mean Landsberg curvature, namely $\mathbf{J}/\mathbf{I}$ is isotropic,*

$$
\mathbf{J} + c(x)F\mathbf{I} = 0, \tag{3.1}
$$

*where $\lambda = \lambda(x)$ and $c = c(x)$ are scalar functions on $M$ if and only if $\mathbf{B} = 0$.*

To prove Theorem 1.2, we need the following key lemma.

**Lemma 3.2.** ([16]) An $(\alpha, \beta)$-metric satisfying $\phi'(0) \neq 0$ is a generalized Berwald manifold if and only if $\beta$ has constant length with respect to $\alpha$.

A Finsler metric $F$ on an $n$-dimensional manifold $M$ is called of isotropic S-curvature, if $\mathbf{S} = (n+1)cF$, where $c = c(x)$ is a scalar function on $M$. In [4], Cheng-Shen characterized $(\alpha, \beta)$-metrics with isotropic S-curvature on a manifold $M$ of dimension $n \geq 3$. Soon, they found that their result holds for the class of $(\alpha, \beta)$-metrics with constant length one-forms, only. In [12], we give a new characterization of the class of generalized Berwald metrics with vanishing S-curvature and prove the following.

**Lemma 3.3.** ([12]) Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be a generalized Berwald $(\alpha, \beta)$-metric on an $n$-dimensional manifold $M$. Suppose that $\phi'(0) \neq 0$. Then $\mathbf{S} = 0$ if and only if $\beta$ is a Killing form with constant length, namely

$$
r_{ij} = 0, \qquad s_j = 0 \tag{3.2}
$$

*Remark* 3.4. Let $\phi = \phi(s)$ be a positive $C^\infty$ function on $(-b_0, b_0)$. For a number $b \in [0, b_0)$, let

$$\Phi := -(Q - sQ')\Big\{n\Delta + 1 + sQ\Big\} - (b^2 - s^2)(1 + sQ)Q'', \tag{3.3}$$

where

$$\Delta := 1 + sQ + (b^2 - s^2)Q'. \tag{3.4}$$

By a direct computation, one can obtain a formula for the mean Cartan torsion of $(\alpha, \beta)$-metrics as follows

$$I_i = -\frac{\Phi}{2\Delta\phi\alpha^2}\Big(\phi - s\phi'\Big)\Big(\alpha b_i - sy_i\Big). \tag{3.5}$$

According to Deickes theorem, a Finsler metric is Riemannian if and only if $\mathbf{I} = 0$. By (3.5), an $(\alpha, \beta)$-metric $F = \alpha\phi(s)$ is Riemannian if and only if $\Phi = 0$.

In [3], Cheng consider regular $(\alpha, \beta)$-metrics with isotropic S-curvature and prove the following.

**Theorem 3.5.** ([3]) *A regular $(\alpha, \beta)$-metric $F := \alpha\phi(\beta/\alpha)$, of non-Randers type on an $n$-dimensional manifold $M$ is of isotropic S-curvature, $\mathbf{S} = (n+1)\sigma F$, if and only if $\beta$ satisfies $r_{ij} = 0$ and $s_j = 0$. In this case, $\mathbf{S} = 0$, regardless of the choice of a particular $\phi = \phi(s)$.*

Now, we are ready to consider generalized Berwald $(\alpha, \beta)$-metrics with isotropic S-curvature and prove the following.

**Lemma 3.6.** *Let $F = \alpha\phi(s)$ , $s = \beta/\alpha$, be an non-Riemannian generalized Berwald $(\alpha, \beta)$-metric on manifold $M$ such that $\phi \neq c_1\sqrt{1 + c_2 s^2} + c_3 s$ for any constant $c_1 > 0$, $c_2$. Then $\mathbf{S} = (n+1)\lambda F$ and $\mathbf{J} = 0$ if and only if $\mathbf{B} = 0$, where $\lambda = \lambda(x)$ is a scalar function on $M$.*

*Proof.* According to the definition of generalized Berwald metrics, a generalized Berwald $(\alpha, \beta)$-metric $F = \alpha\phi(s)$ , $s = \beta/\alpha$, is regular. Then, by Lemma 3.5, we have $\mathbf{S} = 0$.

In [6], Li-Shen found the mean Landsberg curvature of an $(\alpha, \beta)$-metric $F = \alpha\phi(s)$, $s = \beta/\alpha$, as follows

$$\begin{aligned} J_i = &- \frac{1}{\alpha^2\Delta(b^2 - s^2)}\Big[\frac{\Phi}{\Delta} + (n+1)(Q - sQ')\Big](r_0 + s_0)h_i \\ &- \frac{h_i}{2\alpha^3\Delta(b^2 - s^2)}\Big(\Psi_1 + s\frac{\Phi}{\Delta}\Big)\Big(r_{00} - 2\alpha Q s_0\Big) - \frac{\Phi}{2\alpha^3\Delta^2}\Bigg[\alpha Q(\alpha^2 s_i - y_i s_0) \\ &-\alpha Q' s_0 h_i + \alpha^2\Delta s_{i0} + \alpha^2(r_{i0} - 2\alpha Q s_0) - (r_{00} - 2\alpha Q s_0)y_i\Bigg]. \end{aligned} \tag{3.6}$$

where

$$h_i := \alpha b_i - sy_i$$

and

$$\Psi_1 := \sqrt{b^2 - s^2}\Delta^{\frac{1}{2}}\left[\frac{\sqrt{b^2 - s^2}}{\Delta^{\frac{3}{2}}}\right]'.$$

By (3.2) and (3.6) we have:

$$J_i = -\frac{\Phi}{2\alpha\Delta} s_{i0}. \tag{3.7}$$

Considering (3.7) and the assumption $\mathbf{J} = 0$, we obtain

$$s_{ij} = 0. \tag{3.8}$$

Since $r_{ij} = 0$, then (3.8) tell us that $\beta$ is parallel with respect to $\alpha$ and $F$ is a Berwald metric. □

**Proof of Theorem 3.1:** Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be an $(\alpha, \beta)$-metric with relatively isotropic mean Landsberg curvature. The following holds

$$J_k + cFI_k = 0. \tag{3.9}$$

The following holds

$$J_i b^i = -\frac{\Delta}{2\alpha^2}\Big[(r_{00} - 2\alpha Q s_0)\Psi_1 + \alpha(r_0 + s_0)\Psi_2\Big]. \tag{3.10}$$

where

$$\Psi_2 := 2(n+1)(Q - sQ') + 3\frac{\Phi}{\Delta}.$$

By assumption, $F$ has vanishing S-curvature. Then, (3.2) and (3.10) imply that

$$b_i J^i = 0. \tag{3.11}$$

Considering (3.11) and multiplying (3.9) with $b^k$ gives us

$$c(b^k I_k) = 0. \tag{3.12}$$

Let $c \neq 0$, $\forall x \in M$. By (3.12), we get

$$b^k I_k = 0.$$

In this case, (3.5) implies that

$$\frac{\Phi}{2\Delta\phi\alpha^3}(\phi - s\phi')(b^2\alpha^2 - \beta^2) = 0. \tag{3.13}$$

Considering (3.13), one can get $\Phi = 0$ or $\phi - s\phi' = 0$. By (3.5) it follows that $\mathbf{I} = 0$ and then $F$ reduces to a Riemannian metric, which contradicts with the assumption. Thus, we have $c = 0$. Putting it in (3.9) yields $\mathbf{J} = 0$. By Lemma 3.6, $F$ is a Berwald metric. This completes the proof. □

**Proof of Corollary 1.3:** Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be a non-Randers type $(\alpha, \beta)$-metric on manifold $M$ of dimension $n \geq 3$ such that $\phi'(0) \neq 0$. In [2], it is proved that $F$ satisfies $\mathbf{E} = 0$ if and only if $\beta$ is a killing 1-form with constant length with respect to $\alpha$. By Theorem 1.2, we get the proof. □

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