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Research Paper

INTRINSIC PROPERTIES OF FINSLER SPACE WITH A CUBIC CHANGE IN INFINITE SERIES METRIC

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ABSTRACT

In this paper, we investigate a Finsler space characterized by a cubic changed infinite series metric given by $F(\gamma,\beta) = \frac{\beta^2}{(\beta-\gamma)}$. We derive the fundamental tensors necessary to describe the geometric properties of this Finsler space. Additionally, we determine the conditions under which this Finsler space with the cubic modified infinite series metric can be simplified into special types of Finsler spaces, including quasi-C-reducible, semi-C-reducible, C-reducible, and C2-like Finsler spaces, based on its various forms of the Cartan tensor.

1. Introduction

The concept of an (α,β)-metric was introduced by M. Matsumoto [5] in 1972. This metric considers a homogeneous function of degree one with two variables: α, which acts as a Riemannian metric, and and β, as a one-form metric. The Finsler space equipped with the

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(α, β) structure has since been extensively studied by various researchers [3, 6, 8, 9, 10, 18, 19]. In 2004, Lee and Park [4] extended this concept by introducing an r-th series (α, β)-metric

$$L(\alpha, \beta) = \beta \sum_{k=0}^{r} (\frac{\alpha}{\beta})^k, \qquad \alpha < \beta \tag{1.1}$$

The metric function described above is a homogeneous function of degree one in α and β. This function is significant because it simplifies to various important forms of an (α, β)-metric for various value of r. For example:

(1) If we consider $r = 1$ then r-th series metric reduces in a special and important form which is known as Randers metric and given by $L = \alpha + \beta$ which is widely used in the field of physics.
(2) If we consider $r = 2$ then r-th series metric reduces in $L = \alpha + \beta + \frac{\alpha^2}{\beta}$ which represent a combination of Randers metric and Kropina metric.
(3) If we consider $r = \infty$ then r-th series metric can be expressed as $L(\alpha, \beta) = \frac{\beta^2}{\beta - \alpha}$ which is an remarkable form of an (α, β) that represent the difference of Randers and Matsumoto metric and later on it is known as infinite series metric.

Furthermore, they determined the conditions under which a Finsler space with an infinite series metric becomes a Berwald space, Douglas space, or projectively flat. They also explored the conditions required for a two-dimensional Finsler space F^2 with an infinite series metric to be a Landsberg space. Additionally, the differential equations governing the geodesics of F^2 were discussed. Recently, other researchers [17, 18] have examined the significance of the infinite series metric in various contexts.

Shimada's pioneering work on m-th root metrics [15] has been applied in biology as an ecological metric [1]. This theory represents a direct extension of Riemannian metrics, with the second root metric corresponding to a conventional Riemannian metric. The third and fourth root metrics are termed the cubic metric and quartic metric, respectively. Recent advancements underscore the profound impact of m-th root Finsler metrics in fields such as physics, spacetime theory, gravitation, general relativity, and seismic ray theory. Further in 1979, M. Matsumoto [7] introduced the concept of a cubic metric on a differentiable manifold with local coordinates x^i, defined by

$$L(x, y) = (a_{ijk}(x) y^i y^j y^k)^{\frac{1}{3}}$$

where $a_{ijk}(x)$ are components of a symmetric tensor field of $(0, 3)$-type depending on the position co-ordinate x alone, y^i represents direction co-ordinates and a Finsler space with a cubic metric is called the cubic Finsler space. Several authors [1, 2, 11] have extensively examined the m^{th}-root metric and cubic metric, highlighting their significance in the realm of Finsler geometry. In 2011, Pandey and Chaubey [13] introduced the concept of a (γ, β)-metric, where $\gamma = (a_{ijk}(x) y^i y^j y^k)^{\frac{1}{3}}$ represents a cubic metric and $\beta = b_i(x) y^i$ denotes a one-form. Pandey and Chaubey [12, 14] conducted a detailed study on the (γ, β)-metric, exploring significant geometric properties of Finsler spaces equipped with this metric. Concurrently in the same year, Shukla and Mishra [16] investigated the reducibility of the Cartan torsion tensor and curvature in various forms such as quasi C-reducible, semi C-reducible, C-reducible, C2-like, and S3-like properties for Finsler spaces endowed with the (γ, β)-metric.

In this paper, we introduce a Finsler metric by substituting the Riemannian $\alpha = (a_{ij}(x)y^i y^j)^{\frac{1}{2}}$ metric with a cubic Finsler metric $\gamma = (a_{ijk}(x)y^i y^j y^k)^{\frac{1}{3}}$ in the infinite series metric, which is expressed as follows:

$$F(\gamma, \beta) = \frac{\beta^2}{(\beta - \gamma)} \tag{1.2}$$

and refer to it as the cubic changed Infinite series metric. Furthermore, we derive the fundamental tensors required to describe the geometric properties of this Finsler space in Propositions 2.1, 2.5, 2.6 and 2.8. respectively. We also identify the conditions under which the Finsler space with the cubic changed infinite series metric can be reduced to specific types of Finsler spaces, such as quasi-C-reducible, semi-C-reducible, C-reducible, and C2-like Finsler spaces, based on different forms of the Cartan tensor in Theorems 3.3, 3.6, 3.8, and 3.10 respectively.

2. Fundamental tensors for Finsler space with the cubic changed Infinite series metric

In this section, we derive the fundamental metric tensors for the Finsler space with the cubic changed infinite series metric.

Differentiating equation (1.2) partially with respect to y^i we have

$$l_i = \dot{\partial}_i F = \frac{\beta^2}{\gamma^2(\beta - \gamma)^2} a_i + \frac{\beta^2 - 2\gamma\beta}{(\beta - \gamma)^2} b_i \tag{2.1}$$

where $a_{ijk} y^j y^k = a_i(x, y)$.
Equation (2.1) can also be written as

$$y_i = F l_i = F \dot{\partial}_i F = p_{-1} a_i + F F_\beta b_i \tag{2.2}$$

where

$$p_{-1} = \frac{\beta^4}{\gamma^2(\beta - \gamma)^3}, \qquad F F_\beta = \frac{\beta^2(\beta^2 - 2\gamma\beta)}{(\beta - \gamma)^3}$$

Again, differentiating (2.2) with respect to y^j, we get the angular metric tensor $h_{ij} = F \dot{\partial}_i \dot{\partial}_j F$ as

$$h_{ij} = p_{-1} a_{ij} + q_0 b_i b_j + q_{-2}(a_i b_j + a_j b_i) + q_{-4} a_i a_j \tag{2.3}$$

where $2a_{ijk} y^k = a_{ij}(x, y)$, $q_0 = F F_{\beta\beta} = \frac{2\beta^2\gamma^2}{(\beta - \gamma)^4}$, $q_{-2} = \frac{F F_{\beta\gamma}}{\gamma^2} = \frac{-2\beta^3}{\gamma(\beta - \gamma)^4}$ and $q_{-4} = \frac{F}{\gamma^4}(F_{\gamma\gamma} - \frac{2F_\gamma}{\gamma}) = \frac{4\gamma\beta^4 - 2\beta^5}{\gamma^5(\beta - \gamma)^4}$

Owing to the homogeneity or $h_{ij} y^j = 0$, we have two identities,

$$\begin{cases} p_{-1} + q_{-2}\beta + q_{-4}\gamma^3 = 0 \\ q_0 \beta + q_{-2}\gamma^3 = 0 \end{cases} \tag{2.4}$$

Remark: In equation (2.3) the subscripts of coefficients $p_{-1}, q_0, q_{-2}, q_{-4}$ are used to indicate respective degrees of homogeneity of the scalars.

Now, the fundamental metric tensor $g_{ij}(x, y)$ for a Finsler metric is given by

$$g_{ij} = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j} = h_{ij} + l_i l_j$$

therefore,

$$g_{ij} = h_{ij} + l_i l_j = p_{-1} a_{ij} + p_0 b_i b_j + p_{-2}(a_i b_j + a_j b_i) + p_{-4} a_i a_j \tag{2.5}$$

where $p_0 = q_0 + F_\beta^2 = \frac{\beta^4 - 4\gamma\beta^3 + 6\gamma^2\beta^2}{(\beta-\gamma)^4}$, $p_{-2} = q_{-2} + \frac{F_\gamma F_\beta}{\gamma^2} = \frac{\beta^4 - 4\gamma\beta^3}{\gamma^2(\beta-\gamma)^4}$ and $p_{-4} = q_{-4} + \frac{F_\gamma^2}{\gamma^4} = \frac{5\gamma\beta^4 - 2\beta^5}{\gamma^5(\beta-\gamma)^4}$.

Now using above values of p_0, p_{-2}, p_{-4} and (2.4), we get

$$\begin{cases} p_0\beta + p_{-2}\gamma^3 = F F_\beta \\ p_{-2}\beta + p_{-4}\gamma^3 = 0 \end{cases} \tag{2.6}$$

Proposition 2.1. *The normalized supporting element l_i, angular metric tensor h_{ij}, and fundamental metric tensor g_{ij} for the Finsler space with cubic changed infinite series metric are given by equations (2.1), (2.3) and (2.5) respectively.*

Proposition 2.2. *The coefficients q_0, q_{-2}, q_{-4} of the angular metric tensor h_{ij} in the Finsler space with a cubic changed infinite series metric adhere the relation given by equation (2.4).*

Proposition 2.3. *The coefficients p_0, p_{-2}, p_{-4} of the fundamental metric tensor g_{ij} in the Finsler space with a cubic changed infinite series metric adhere the relation given by equation (2.6).*

Since we know that

Proposition 2.4. *Let a non-singular symmetric n-matrix (A_{ij}) and n quantities c_i be given and put $B_{ij} = A_{ij} + c_i c_j$. The inverse matrix (B^{ij}) of (B_{ij}) and the det(B_{ij}) are given by,*

$$B^{ij} = A^{ij} - \frac{1}{1+c^2} c^i c^j, \; det(B_{ij}) = A(1+c^2)$$

where (A^{ij}) is the inverse matrix of (A_{ij}), $A = det(A_{ij})$, $c^i = A^{ij} c_j$ and $c^2 = c^i c_i$

Now from (2.5), the component g_{ij} can be written as

$$g_{ij} = p_{-1} a_{ij} + c_i c_j + d_i d_j$$

where we considered
$c_i = \pi b_i, \; d_i = \pi_0 b_i + \pi_{-2} a_i, \; \pi^2 + \pi_0^2 = p_0, \; \pi_0 \pi_{-2} = p_{-2}, \pi_{-2}^2 = p_{-4}$

Then putting, $B_{ij} = p_{-1} a_{ij} + c_i c_j$, we have, $g_{ij} = B_{ij} + d_i d_j$

Using the properties of the Kronecker-delta and tensor algebra, we have $B_{ij} B^{jk} = \delta_i^k$
Then,

$$B^{ij} = \frac{1}{p_{-1}} a^{ij} - \frac{c^i c^j}{p_{-1}(p_{-1} + c^2)}$$

where a^{ij} is the reciprocal of a_{ij}, $c^i = a^{ij} c_j$
Now by using proposition 2.1, we have

$$g^{ij} = B^{ij} - \frac{d^i d^j}{1+d^2} \tag{2.7}$$

where $d_i = B_{ij}d^j, d^2 = d_i d^i$

$|g_{ij}| = |B_{ij}|(1+d^2) = |p_{-1}a_{ij}|\frac{(p_{-1}+c^2)}{p_{-1}}(1+d^2) = p_{-1}^{n-1}a(p_{-1}+c^2)(1+d^2)$

where a is the determinant of a_{ij}.

$$g^{ij} = \frac{1}{p_{-1}}a^{ij} - \frac{c^i c^j}{p_{-1}(p_{-1}+c^2)} - \frac{d^i d^j}{1+d^2}$$

Now,

$$d^i = \frac{1}{p_{-1}}\left[\left(\frac{\pi_0 p_{-1} - \pi^2\pi_{-2}\bar{a}}{p_{-1}+c^2}\right)B^i + \pi_{-2}a^i\right]$$

where

$a^{ij}b_j = B^i$, $a^{ij}a_j = a^i$, $B^i b_i = b^2 = a^{im}b_m b_i, a_i B^i = a^{im}a_i b_m = a^i b_i = \bar{a}, \pi^2 b^2 = c_i c^i = c^2$, $a_i a^i = a^2$ and $a^{ij}(x,y)$ is the inverse matrix of $a_{ij}(x,y)$.

Again,

$d^i d^j = \frac{1}{p_{-1}^2}\left[\frac{(\pi_0 p_{-1} - \pi^2\pi_{-2}\bar{a})^2}{(p_{-1}+c^2)^2}B^i B^j + \frac{(p_{-1}p_{-2} - \pi^2 p_{-4}\bar{a})}{(p_{-1}+c^2)}(B^i a^j + B^j a^i) + p_{-4}a^i a^j\right]$

Now,

$$d^2 = d^i d_i = \frac{1}{p_{-1}(p_{-1}+c^2)}\left[\pi_0^2 b^2 p_{-1} + 2p_{-1}p_{-2}\bar{a} - \pi^2 p_{-4}\bar{a}^2 + p_{-4}p_{-1}a^2 + p_{-4}c^2a^2\right]$$

Again,

$$|g_{ij}| = p_{-1}^{n-1}a(p_{-1}+c^2)(1+d^2) = p_{-1}^{n-2}a\tau_{-2}$$

where, $\tau_{-2} = p_{-1}(p_{-1} + p_0 b^2 + p_{-2}\bar{a}) + (p_{-2}p_{-1}\bar{a} - \pi^2 p_{-4}\bar{a}) + p_{-4}p_{-1}a^2 + p_{-4}c^2a^2$

Thus the reciprocal of g^{ij} of g_{ij} is given by

$$g^{ij} = \frac{1}{p_{-1}}a^{ij} - S_2 B^i B^j - S_0(B^i a^j + B^j a^i) - S_{-2}a^i a^j \tag{2.8}$$

where $S_0 = \frac{\beta^4\gamma - 4\gamma^2\beta^3 - \pi^2\bar{a}(11\gamma\beta^3 - 21\beta^2\gamma^2 + 17\beta\gamma^3 - 5\gamma^4 - 2\beta^4)}{\tau_{-2}\gamma^3(\beta-\gamma)^4}$

$S_{-2} = \frac{5\gamma\beta^4 + 11\gamma^3\beta^3c^2 - 21\beta^2\gamma^4c^2 + 17\beta\gamma^5c^2 - 5\gamma^6c^2 - 2\beta^5 - 2\beta^4\gamma^2c^2}{\tau_{-2}\gamma^5(\beta-\gamma)^4}$

$S_2 = \frac{\pi_0^2(\beta^9\gamma - \beta^8\gamma^2) + \pi^2\tau_{-2}\gamma^5(\beta-\gamma)^7 + \pi^4\bar{a}^2(5\gamma\beta^4 - 2\beta^5)(\beta-\gamma)^3 - \pi^2\bar{a}\gamma(2\beta^8 - 8\gamma\beta^7)}{\tau_{-2}(\beta-\gamma)(\gamma\beta^8 + c^2\gamma^3\beta^4(\beta-\gamma)^3)}$.

Proposition 2.5. *The reciprocal g^{ij} of the fundamental metric tensor g_{ij} for the cubic changed infinite series metric is given by equation (2.8).*

Now differentiating (2.5) with respect to y^k, we get

$$2C_{ijk} = 2p_{-1}a_{ijk} + p_{0\beta}b_i b_j b_k + \frac{p_{-4\gamma}}{\gamma^2}a_i a_j a_k + \Pi_{(ijk)}\left(P_i a_{jk} + p_{-2\beta}a_i b_j b_k + \frac{p_{-2\gamma}}{\gamma^2}a_i a_j b_k\right) \tag{2.9}$$

where $\Pi_{(ijk)}$ represents the sum of cyclic permutations of i, j, k and

$$P_i = p_{-4}a_i + p_{-2}b_i \tag{2.10}$$

or,

(2.11) $2p_{-1}C_{ijk} = 2p_{-1}^2 a_{ijk} + r_{-2}b_ib_jb_k + \Pi_{(ijk)}(P_ih_{jk} + r_{-4}a_ib_jb_k + r_{-6}a_ia_jb_k) + r_{-8}a_ia_ja_k$

where $r_{-2} = \frac{12\gamma^3\beta^5 - 6\gamma^2\beta^6}{\gamma^2(\beta-\gamma)^8}$, $r_{-4} = \frac{-15\gamma^4\beta^6 + 10\gamma^2\beta^8 - 2\gamma^3\beta^7 - 2\gamma\beta^9}{\gamma^6(\beta-\gamma)^8}$ $r_{-6} = \frac{-6\gamma\beta^8 - 40\gamma^2\beta^7 - 4\beta^9}{\gamma^7(\beta-\gamma)^8}$ and $r_{-8} = \frac{34\gamma^2\beta^8 - 2\gamma\beta^9 - 2\beta^{10}}{\gamma^{10}(\beta-\gamma)^8}$.

Thus

Proposition 2.6. *The Cartan torsion tensor C_{ijk} for the Finsler space with cubic changed infinite series metric are given by equation (2.11).*

Proposition 2.7. *The coefficients $r_{-8}, r_{-6}, r_{-4}, r_{-2}$ of the Cartan torsion tensor in the Finsler space with a cubic changed infinite series metric adhere to the following relation:*

(2.12)
$$r_{-\mu}\beta + r_{-\mu-2}\gamma^3 = 0, \quad \mu = 2, 4, 6.$$

Using (2.4), (2.6) and (2.12), we have

(2.13)
$$p_{-4} = \phi p_{-2}, \quad r_{-\mu-2} = \phi^{\frac{\mu}{2}} r_{-2}, \qquad \mu = 2, 4, 6.$$

where, $\phi = -\frac{\beta}{\gamma^3}$
Using (2.13) in (2.11), we easily get

(2.14)
$$2p_{-1}C_{ijk} = 2p_{-1}^2 a_{ijk} + \Pi_{(ijk)}(H_{jk}P_i)$$

where
$$H_{jk} = h_{jk} + \frac{r_{-2}}{3p_{-2}^3} P_jP_k$$

Further, by direct computation from C_{ijk} and g^{ij}, we have,

(2.15)
$$C_i = p_{-1}A_i + Aa_i + Bb_i$$

where A and B are certain scalar.

Proposition 2.8. *The Mean Cartan tensor C_i for the Finsler space with cubic changed infinite series metric are given by (2.15).*

3. Reducibility of Cartan Tensor for the Finsler space with cubic changed infinite series metric

In this section, we will explore the conditions under which the Finsler space with a cubic changed infinite series metric can be classified as a C-reducible Finsler space, Semi C-reducible Finsler space, quasi C-reducible Finsler space, and C2-like Finsler space.

Definition 3.1. A Finsler space (M^n, F) with dimension $n \geq 3$ is said to be quasi C-reducible if the Cartan tensor C_{ijk} can be written in the form

(3.1)
$$C_{ijk} = Q_{ij}C_k + Q_{jk}C_i + Q_{ki}C_j$$

where Q_{ij} is a symmetric indicatory tensor and $C_i = C_{ijk}g^{jk}$ is the torsion vector.

Now, from (2.10) and (2.15)

$$2p_{-1}C_{ijk} = 2p_{-1}^2 a_{ijk} + \Pi_{(ijk)}\left(H_{jk}P_i\right)$$
$$2p_{-1}C_{ijk} = 2p_{-1}^2 a_{ijk} + \Pi_{(ijk)}\left\{H_{jk}\left(p_{-4}a_i + p_{-2}b_i\right)\right\}$$

Using (2.15) we have,

$$2p_{-1}C_{ijk} = 2p_{-1}^2 a_{ijk} + \Pi_{(ijk)}\left\{H_{jk}\left(\frac{p_{-4}}{A}\left(C_i - p_{-1}a_i - Bb_i\right) + p_{-2}b_i\right)\right\}$$

$$2p_{-1}C_{ijk} = 2p_{-1}^2 a_{ijk} + \Pi_{(ijk)}\left\{\frac{p_{-4}}{A}H_{jk}C_i - \frac{p_{-1}p_{-4}}{A}H_{jk}A_i - \left(\frac{p_{-4}B}{A} - p_{-2}\right)H_{jk}b_i\right\}$$

$$C_{ijk} = p_{-1}a_{ijk} - \Pi_{(ijk)}\left\{\left(\frac{p_{-4}A_i}{2A} + \left(\frac{p_{-4}B}{2p_{-1}A} - \frac{p_{-2}}{2p_{-1}}\right)b_i\right)H_{jk}\right\} + \frac{p_{-4}}{2p_{-1}A}\Pi_{(ijk)}\left(H_{jk}C_i\right)$$

Thus we have,

Lemma 3.2. *In a Finsler space with cubic changed infinite series metric, the Cartan tensor can be written in the form*

$$C_{ijk} = V_{ijk} + \Pi_{(ijk)}(Q_{jk}C_i) \tag{3.2}$$

where $Q_{jk} = \frac{p_{-4}}{2p_{-1}A}H_{jk}$ *is a symmetric tensor of order two and*

$$V_{ijk} = p_{-1}a_{ijk} - \Pi_{(ijk)}\left\{\left(\frac{p_{-4}A_i}{2A} + \left(\frac{p_{-4}B}{2p_{-1}A} - \frac{p_{-2}}{2p_{-1}}\right)b_i\right)H_{jk}\right\}.$$

Thus by the above lemma and definition (3.1), we have

Theorem 3.3. *A Finsler space with cubic changed infinite series metric is quasi C-reducible if and only if the tensor* V_{ijk} *of equation* (3.2) *vanishes identically and* $\frac{p_{-4}}{2p_{-1}A} \to 1$.

Definition 3.4. A Finsler space (M^n, F) with dimension $n \geq 3$ is said to be semi C-reducible if the Cartan tensor C_{ijk} is written in the form

$$C_{ijk} = \frac{r}{(n+1)}(h_{ij}C_k + h_{ki}C_j + h_{jk}C_i) + \frac{t}{C^2}C_iC_jC_k \tag{3.3}$$

where r and t are scalar function such that $r + t = 1$

From equation (2.14) we have,
$C_{ijk} = p_{-1}a_{ijk} + \frac{1}{2p_{-1}}\Pi_{(ijk)}\left\{h_{jk}P_i\right\} + \frac{r_{-2}}{2p_{-1}p_{-2}^3}P_iP_jP_k.$
Using (2.15), we have

$$\begin{aligned}\frac{r_{-2}}{2p_{-1}p_{-2}^3}P_iP_jP_k = &\frac{r_{-2}p_{-4}^3}{2p_{-1}p_{-2}^3A^3}C_iC_jC_k + \frac{r_{-2}}{2p_{-1}p_{-2}^3}\Pi_{(ijk)}\left[-\frac{p_{-4}^3p_{-1}}{A^3}(C_iC_jA_k)\right.\\ &- \frac{p_{-4}^2}{A^2}\left(\frac{p_{-4}\beta}{A} - p_{-2}\right)(C_iC_jb_k) + \frac{p_{-4}^3p_{-1}^2}{A^3}(C_iA_jA_k)\\ &+ \frac{p_{-4}^2p_{-1}}{A^2}\left(\frac{p_{-4}\beta}{A} - p_{-2}\right)(C_iA_jb_k)\\ &+ \frac{p_{-4}}{A}\left(\frac{p_{-4}\beta}{A} - p_{-2}\right)^2(C_ib_jb_k)\\ &- \frac{p_{-4}^2p_{-1}^2}{A^2}\left(\frac{p_{-4}\beta}{A} - p_{-2}\right)A_iA_jb_k\\ &\left.-\frac{p_{-4}p_{-1}}{A}\left(\frac{p_{-4}\beta}{A} - p_{-2}\right)^2A_ib_jb_k\right]\\ &- \left(\frac{p_{-4}^3p_{-1}^3}{A^3}A_iA_jA_k + \left(\frac{p_{-4}\beta}{A} - p_{-2}\right)^3b_ib_jb_k\right)\frac{r_{-2}}{2p_{-1}p_{-2}^3}.\end{aligned}$$

Now,

$$C_{ijk} = \frac{p_{-4}}{2p_{-1}A}\Pi_{(ijk)}(h_{jk}C_i) + \frac{r_{-2}p_{-4}^3}{2p_{-1}p_{-2}^3A^3}C_iC_jC_k + U_{ijk} \tag{3.4}$$

where,

$$\begin{aligned}U_{ijk} = p_{-1}a_{ijk} - \Pi_{(ijk)}\frac{p_{-4}}{2p_{-1}A}\left\{p_{-1}A_i + \left(B - \frac{p_{-2}A}{p_{-4}}\right)b_i\right\}h_{jk}+\\
\frac{r_{-2}}{2p_{-1}p_{-2}^3}\Pi_{(ijk)}\Bigg[-\frac{p_{-4}^3p_{-1}}{A^3}C_iC_jA_k\\
-\frac{p_{-4}^2}{A^2}\left(\frac{p_{-4}\beta}{A} - p_{-2}\right)C_iC_jb_k + \frac{p_{-4}^3p_{-1}^2}{A^3}C_iA_jA_k\\
+\frac{p_{-4}^2p_{-1}}{A^2}\left(\frac{p_{-4}\beta}{A} - p_{-2}\right)C_iA_jb_k + \frac{p_{-4}}{A}\left(\frac{p_{-4}\beta}{A} - p_{-2}\right)^2C_ib_jb_k\\
-\frac{p_{-4}^2p_{-1}^2}{A^2}\left(\frac{p_{-4}\beta}{A} - p_{-2}\right)A_iA_jb_k - \frac{p_{-4}p_{-1}}{A}\left(\frac{p_{-4}\beta}{A} - p_{-2}\right)^2A_ib_jb_k\Bigg]\\
-\left(\frac{p_{-4}^3p_{-1}^3}{A^3}A_iA_jA_k + \left(\frac{p_{-4}\beta}{A} - p_{-2}\right)^3b_ib_jb_k\right)\frac{r_{-2}}{2p_{-1}p_{-2}^3}.\end{aligned}$$

Therefore, if $r = \frac{(n+1)p_{-4}}{2p_{-1}A}$, $t = \frac{C^2r_{-2}p_{-4}^3}{2p_{-1}p_{-2}^3A^3}$ and $C_iC^i = C^2$ then above equation can be rewritten as

$$C_{ijk} = \frac{r}{(n+1)}\Pi_{(ijk)}(h_{jk}C_i) + \frac{t}{C^2}C_iC_jC_k + U_{ijk} \tag{3.5}$$

Thus we have,

Lemma 3.5. *In a Finsler space with cubic changed infinite series metric, the Cartan tensor can be rewritten in the form* (3.5).

By the above lemma and equation (3.3) we can conclude

Theorem 3.6. *A Finsler space with cubic changed infinite series metric is semi C-reducible Finsler space if and only if the tensor U_{ijk} vanishes identically and $r + t = 1$.*

Definition 3.7. A Finsler space (M^n, F) with dimension $n \geq 3$ is said to be C-reducible if the Cartan tensor C_{ijk} is written in the form

$$C_{ijk} = \frac{1}{n+1}(h_{ij}C_k + h_{ki}C_j + h_{jk}C_i) \tag{3.6}$$

From (2.14) and (3.6) it follows that

$$\begin{aligned}2p_{-1}C_{ijk} &= 2p_{-1}^2a_{ijk} + \Pi_{(ijk)}(H_{jk}P_i)\\
&= 2p_{-1}^2a_{ijk} + \Pi_{(ijk)}(h_{jk}P_i) + \frac{r_{-2}}{p_{-2}^3}P_iP_jP_k\\
2p_{-1}\left(\frac{1}{n+1}\right)\Pi_{(ijk)}(h_{jk}C_k) &= 2p_{-1}^2a_{ijk} + \Pi_{(ijk)}(h_{jk}P_i) + \frac{r_{-2}}{p_{-2}^3}P_iP_jP_k\\
2p_{-1}^2a_{ijk} + \frac{r_{-2}}{p_{-2}^3}P_iP_jP_k &= 2p_{-1}\left(\frac{1}{n+1}\right)\Pi_{(ijk)}(h_{jk}C_k) - \Pi_{(ijk)}(h_{jk}P_i)\\
&= \Pi_{(ijk)}\left(\frac{2p_{-1}}{(n+1)}C_k - P_k\right)h_{ij}\end{aligned}$$

$$2p_{-1}^2 a_{ijk} + \frac{r_{-2}}{p_{-2}^3} P_i P_j P_k = \Pi_{(ijk)}(h_{ij} N_k) \tag{3.7}$$

where $N_k = \frac{2p_{-1}}{(n+1)} C_k - P_k$. Conversely, if (3.7) is satisfied for certain covariant vector N_k, then from (2.14) we have

$$2p_{-1} C_{ijk} = \Pi_{(ijk)}(h_{ij} N_k) + \Pi_{(ijk)}(h_{ij} P_k)$$

$$2p_{-1} C_{ijk} = \Pi_{(ijk)} \{h_{ij}(N_k + P_k)\} \tag{3.8}$$

which gives (3.6)

Theorem 3.8. *A Finsler space with cubic changed infinite series metric is C-reducible Finsler space if and only if (3.7) holds good.*

Definition 3.9. A Finsler space (M^n, F) with dimension $n \geq 2$ is called C2-like if the Cartan tensor C_{ijk} is written in the form

$$C^2 C_{ijk} = C_i C_j C_k \tag{3.9}$$

From equation (3.5) we have

$$\begin{aligned} C_{ijk} &= \frac{r}{(n+1)} \Pi_{(ijk)}(h_{jk} C_i) + \frac{t}{C^2} C_i C_j C_k + U_{ijk} \\ &= \frac{1}{C^2} \left(\frac{rC^2}{(n+1)} \Pi_{(ijk)}(h_{jk} C_i) + t C_i C_j C_k + C^2 U_{ijk} \right) \\ C^2 C_{ijk} &= t \left[\frac{1}{t} \left(\frac{rC^2}{(n+1)} \Pi_{(ijk)}(h_{jk} C_i) + C^2 U_{ijk} \right) + C_i C_j C_k \right] \end{aligned}$$

$$C^2 C_{ijk} = t(U'_{ijk} + C_i C_j C_k) \tag{3.10}$$

where

$$U'_{ijk} = \frac{1}{t} \left(\frac{rC^2}{(n+1)} \Pi_{(ijk)}(h_{jk} C_i) + C^2 U_{ijk} \right)$$

Theorem 3.10. *A Finsler space endowed with a cubic changed infinite series metric is considered a C2-like Finsler space if the tensor U'_{ijk} from equation (3.10) vanishes completely and $t \to 1$.*

4. Conclusions

In the present paper, we investigated the Finsler space with cubic changed infinite series metric and derived its fundamental tensors: the supporting line element, angular metric tensor, fundamental metric tensor and its reciprocal, Cartan torsion tensor and mean Cartan tensor in Proposition 2.1, 2.5, 2.6 and 2.8. Furthermore, we identified conditions under which the Finsler space with cubic changed infinite series metric can be classified as quasi C-reducible, semi C-reducible, C-reducible, and C2-like Finsler spaces in Theorems 3.3, 3.6, 3.8, and 3.10 respectively..

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