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Research Paper

GAMMA VARIANT OF $(p, q)-$ BERNSTEIN TYPE NOVEL OPERATORS

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ABSTRACT

In this paper, we are concerned with a new modification of the well-known (p,q)-Bernstein novel type operators with the gamma integral functions. The direct results demonstrate several aspects of approximations. Such as the rate of convergence theorem using Peetre's K-functional and Korovkin's theorem, which also validates the well-known Voronovskaja's theorem and the convergence theorem for Lipschitz continuous functions.

1. Introduction

The Bernstein polynomial on the closed interval $[0, 1]$ is a fascinating and well-known polynomial introduced in 1912 by S.N. Bernstein [3]. Such as

$$B_n(f, x) = \sum_{k=0}^{n} b_{n,k}(x) f(\frac{k}{n})$$

where

$$b_{n,k}(x) = \binom{n}{k} x^k (1 - x)^{n-k}.$$

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and when $k = 0, 1, 2.......n$
q-Bernstein polynomials for $f \in C[0,1]$ proposed by G.M. Phillips [19].

$$B_{n,q}(f;x) = \sum_{k=0}^{n} b_{n,k}(q;x) f(\frac{k}{n})$$

where

$$b_{n,k}(q;x) = \binom{n}{k} x^k (1-x)_q^{n-k}.$$

In addition to the novel modification of the q-Bernstein operators and (p,q)-integers on those operators with their limit and the Voronovskaja approximation with some properties, for $0 < q < p \le 1$ for all $f \in C[0,1]$ and $x \in [0,1]$ in ([6],[10],[11]), such that

$$B_{n,p,q}(f;x) = \sum_{k=0}^{n} p^{\{k(k-1)-n(n-1)\}/2} \binom{n}{k}_{p,q} x^k (1-x)_{p,q}^{n-k} f\left(p^n \left(\frac{[k]}{[n]}\right)_{p,q}\right),$$

$\forall\ n \in N$ and if $p = 1$ and $0 < q < 1$, then bring to the point of q-Bernstein operator.

$$B_{n;q}(f;x) = \sum_{k=0}^{n} \binom{n}{k}_{p,q} x^k (1-x)_{p,q}^{n-k} f(\frac{k}{n})$$

A new generalization with copious variations for those operators, like the (p,q)-Bernstein and (p,q)- Durrmeyer operators in 2009 By Gupta and in 2015 was forwarded by Mursaleen et al.([13],[18]). Some approximation properties for q-integers and (p,q)-integers with the convexity of functions offered by Gupta ([12],[15],[16]). A new modification of the Narayana operators using (k,t) bivariate with (p,q) generalized Bernstein operators and their applications in 2024 proposed by Bala and Mishra [1]. New Bernstein-type operators based on beta-modification with a graphical depiction of the newly created operators for $f \in C([0,1))$ are defined as Beta Bernstein operators. For $x \in [0,1]$, and $\beta_n : C[0,1] \to C[0,1]$ was presented by Dhawal J.et al.[7], such as

$$\beta_n(f;x) = \sum_{k=0}^{n} P_{n,k}(x) f(k/n) dt$$

where

$$P_{n,k}(x) = \binom{n}{k} \frac{\beta(nx+k+1, 2n-k-nx+1)}{\beta(nx+1, n-nx+1)}.$$

at here $\beta(a,b)$ is beta function and defined as

$$\beta(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1} dt,$$

where $(a,b) \ge 0$.

The summation integral formula provided by J.L.Durrmeyer furnished for additional generalizations of Bernstein operators in [9], including as

$$D_n(f;x) = \sum_{k=0}^{n} p_{n,k}(x) \int_0^1 p_{n,k}(t) f(t) dt.$$

The famous Szasz Mirakayan operator in [17] such as :

$$S_n(f,x) = \sum_{k=0}^{\infty} s_{n,k}(x) f(k/n) dt$$

where $S_{n,k}(x) = e^{-nx}\frac{(nx)^k}{k!}$. Furthermore, Baskakov introduced an operator known as the Baskakov operator, which is applicable to continuous functions. There are defined examples for $x \in [0, \infty)$, such as $V_n : C[0, \infty) \to C[0, \infty)$ in [2].

$$V_n(x) = \sum_{k=0}^{\infty} v_{n,k}(x) f(k/n),$$

where $v_{n,k}(x) = \binom{n+k-1}{k}\frac{x^k}{(1+x)^{n+k}}$.
There has been a great deal of generalization of Bernstein-type operators available to academics. In 2016 the (p, q) Bernstein-Durrmeyer operators with beta integral using some moments where for each $n \in N$ and $f \in C[0, 1]$ Honey Sharma [20].

$$D_n^{p,q}(f; x) = [n+1]_{p,q} p^{-n^2} \sum_{k=0}^{n} b_{n,k}^{p,q}(x)(\frac{q}{p})^{-k} \int_0^1 b_{n,k}^{p,q}(qt) f(t) d_{p,q}t,$$

for $p > 1$ then (p, q)-Bernstein Durrmeyer operators with beta integral.

Definition 1.1. ([20]). If $0 < p < q \leq 1$ and for every $s, t \in R^+$, then the (p, q)-beta integral is

$$\beta_{p,q}(t, s) = \int_0^1 x^{t-1}(1 - qx)_{p,q}^{s-1} d_{p,q}(x)$$

and proposed a connection between q-beta and (p, q)-beta integrals.

Let $0 < p < q \leq 1$ then (p, q) integer $[n]_{p,q}!$ such as

$$[n]_{p,q}! = \frac{p^n - q^n}{p - q}$$

,

$$[n]_{p,q}! = [1]_{p,q}, [2]_{p,q}, \qquad ......[n]_{p,q}.$$

If for every $n \geq 1$ and $[n]_{p,q}! = 1$ if $n = 0$ for special case with integers $0 \leq k \leq n$ such as

$$\binom{n}{k}_{p,q} = \frac{[n]_{p,q}!}{[k]_{p,q}![n-k]_{p,q}!}$$

and expansion of (p, q)-polynomials is

$$(x + y)_{p,q}^n = (x + y)\{(px + qy)(p^2x + q^2y)..........(p^{n-1}x + q^{n-1}y)\}.$$

If $f : [0, a] \to R$ then integration of $f(x)$ is defined by

$$\int_0^a f(x) d_{p,q}(x) = (p - q)a \sum_{k=0}^{\infty} \frac{q^k}{p^{k+1}} f(\frac{q^k}{p^{k+1}} a)$$

when $|\frac{p}{q}| > 1$

In 2022 by Cai et al. dealt with the new generalization of beta Bernstein with test functions, uniform convergence, the Peetre K-functional, and functions of the Lipschitz class [5], such as

$$\tilde{B_m}(K : x) = \frac{1}{m} \sum_{p=0}^{m} \binom{m}{p} (p - mx)^2 x^{p-1}(1-x)^{m-p-1} \frac{1}{\beta(p+1, m-p+1)} \int_0^1 u^p(1-u)^{m-p} K(u) du.$$

for every $x \in (0, 1), m \in N$ and $\beta(p + 1, m - p + 1)$ is a beta function and if $r, s > 0$ then

$$\beta_{r,s} = \int_0^1 x^{r-1}(1 - x)^{s-1} dx$$

Numerous mathematicians have proposed several Bernstein generalizations after studying a new hybrid method for data analysis suggested by Vajargah and Nouraldin [4]. We provide a new variation of the (p,q)-Bernstein operators defined by a gamma generalization.

$$B_n^{p,q}(f;x) = \sum_{k=0}^{n} b_{n,k}^{p,q}(f,x) f(\frac{k}{n})_{p,q}. \tag{1}$$

where

$$b_{n,k}^{p,q}(f;x) = q^k \binom{n}{k}_{p,q} p^n e^{-nx}[nx]^k,$$

If for every $n \in N$ and $p = 1$, $0 < q < 1$, then the above equation reduces to the q-Bernstein operator.

$$b_{n,k}^{q}(f;x) = q^k \binom{n}{k}_{q} e^{-nx}[nx]^k, \tag{2}$$

In the paper, we gave all estimates according to equation (2) because it's a special case of well-known Bernstein operators. In 2024 a novel Stancu-type adaptation of Bernstein-Kantorovich bivariate operators with an exponential class was created. They also provided some well-known theorems and approximation properties by Kanat and Su [21].

2. Main Results

Moments of the (p,q)-Bernstein operators

We now infer a few moments of those altered operators.

Lemma 2.1. *If* $e_i(t) = t^i$, $i = 0, 1, 2$ *and for* $x \in [0, \infty]$ *and* $n \in N$ *then*

- $B_n^{p,q}(e_0;x) \equiv B_n(t^0, x) = B_n(1, x) = e^{-nx}$
- $B_n^{p,q}(e_1;x) \equiv B_n(t^1, x) = B_n(t, x) = qxe^{-nx}$
- $B_n^{p,q}(e_2;x) \equiv B_n(t^2, x) = e^{-nx}qx\left(\frac{1}{n} + qxn(n-1)\right)$

Proof. Let $i = 0$ in the above statement then we get with using the equations (1) and (2) where

$$b_{n,k}^{p,q}(f;x) = q^0 \binom{n}{0}_{p,q} e^{-nx}[nx]^0 = e^{-nx},$$

$$B_n(t^0, x) = B_n(1, x) = e^{-nx}$$

And

$$B_n(t^1, x) = B_n(t, x) = \sum_{k=0}^{n} b_{n,k}(f, x)(\frac{k}{n}) =$$

$$\sum_{k=0}^{n} b_{n,k}^{p,q}(f;x)(\frac{k}{n}) + \sum_{k=1}^{n} b_{n,k}^{p,q}(f;x)(\frac{k}{n}) =$$

$$q^0 \binom{n}{0}_{p,q} e^{-nx}[nx]^0(\frac{0}{n}) + q^1 \binom{n}{1}_{p,q} e^{-nx}[nx]^1 \frac{1}{n} = qxe^{-nx},$$

$$B_n(t, x) = qxe^{-nx}$$

The next moment is the, where

$$B_n(t^2,x) = B_n(t^2,x) = \sum_{k=0}^{n} b_{n,k}(f,x)(\frac{k}{n})^2$$

$$= \sum_{k=0}^{n} b_{n,k}^{p,q}(f;x)(\frac{k}{n})^2 + \sum_{k=1}^{n} b_{n,k}^{p,q}(f;x)(\frac{k}{n})^2 + \sum_{k=2}^{n} b_{n,2}^{p,q}(f;x)(\frac{k}{n})^2$$
$$= q^0 \binom{n}{0}_{p,q} e^{-nx}[nx]^0(\frac{0}{n})^2 + q^1 \binom{n}{1}_{p,q} e^{-nx}[nx]^1(\frac{1}{n})^2 + q^2 \binom{n}{2}_{p,q} e^{-nx}[nx]^2(\frac{2}{n^2})$$

$$= qnxe^{-nx}(\frac{1}{n^2}) + q^2\frac{n(n-1)}{2}(e^{-nx})[nx]^2(\frac{2}{n^2}),$$

$$B_n(t^2,x) = e^{-nx}(\frac{qx}{n}) + q^2e^{-nx}n(n-1)$$
$$= e^{-nx}qx\left(\tfrac{1}{n} + qxn(n-1)\right).$$ □

2.1. Central moments of above Bernstein operators.

Lemma 2.2. *If $x \in [0,1]$ and for $0 < q < p \leq 1$, using the above moments of the lemma, then*

(1) $B_n(t-x;x) = e^{-nx}x(q-1)$
(2) $B_n((t-x)^2;x) = e^{-nx}x\left(\tfrac{q}{n} + n(n-1)xq^2 - 2xq + x\right)$

Proof.

(1) $$B_n(t-x;x) = B_n(t,x) - xB_n(1,x) = e^{-nx}qx - e^{-nx}x$$
$$= e^{-nx}x(q-1)$$

(2) $$B_n((t-x)^2;x) = B_n(t^2,x) - 2xB_n(t,x) + x^2B_n(1,x)$$
$$= e^{-nx}\left(\frac{1}{n} + n(n-1)(qx)\right) - 2xe^{-nx}qx + e^{-nx}x^2$$
$$= e^{-nx}x\left(\frac{q}{n} + n(n-1)xq^2 - 2xq + x\right)$$
$$= \Phi(x).$$

□

3. Convergence theorem for Bernstein operators

Theorem 3.1. *If a function $f \in C[0,1]$ for every $\epsilon > 0$ then there is an existence of N such that*

$$|f(x) - B_n^{p,q}(f;x)| < \epsilon,$$

for all $x \in [0,1]$ and $n \geq N$.

Proof. We know that the inequality

$$\left(\frac{k}{n}-x\right)^2 = (\frac{k}{n})^2 - 2(\frac{k}{n})x + x^2 \qquad (3)$$

both sides of equation (3) with the sum of k = 0 to n, then

$$\sum_{k=o}^{n}\left(\frac{k}{n}-x\right)^2\binom{n}{k}q^k e^{-nx}[nx^k]$$

$$= B_n^{p,q}(t^2;x) - 2xB_n^{p,q}(t;x) + x^2B_n^{p,q}(1;x)$$

$$= B_n^{p,q}\left((t-x)^2;x\right) = e^{-nx}x\left(\frac{q}{n} + n(n-1)xq^2 - 2xq + x\right)$$

By using above lemma

$$\sum_{k=o}^{n}\left(\frac{k}{n}-x\right)^2\binom{n}{k}q^k e^{-nx}[nx^k] = e^{-nx}x\left(\frac{q}{n} + n(n-1)xq^2 - 2xq + x\right)$$

Now we select a number $\delta > 0$ and if S_δ is a set for all values of k and holds $|\frac{k}{n} - x| \geq \delta$ then

$$\frac{1}{\delta^2}\left(\frac{k}{n} - x^2\right) \geq 1$$

hence

$$\Sigma_{k\in S_\delta}\binom{n}{k}q^k e^{-nx}[nx]^k \leq \frac{1}{\delta^2}\Sigma_{k\in S_\delta}\left(\frac{k}{n}-x\right)^2\binom{n}{k}q^k e^{-nx}[nx]^k$$

Since $0 \leq e^{-nx}.qx \leq \frac{1}{2}$ on $[0,1]$ then we get

$$\sum_{k=0}^{n}\binom{n}{k}q^k e^{-nx}[nx]^k \leq \frac{1}{2n\delta^2} \qquad (4)$$

then we can write

$$\sum_{k=0}^{n} = \sum_{k\epsilon S_\delta} + \sum_{k\notin S_\delta}$$

Now we can write the difference between $f(x)$ and $B_n^{p,q}(f;x)$ we have

$$f(x) - B_n^{p,q}(f;x) = \sum_{k=0} n\left(f(x) - f\frac{k}{n}\right)\binom{n}{k}q^k e^{-nx}[nx]^k$$

and so

$$f(x) - B_n^{p,q}(f;x) = \sum_{k\in S_\delta}\left(f(x) - f(\frac{k}{n})\right)\binom{n}{k}q^k e^{-nx}[nx]^k +$$

$$\sum_{k\notin S_\delta}\left(f(x) - f(\frac{k}{n})\right)\binom{n}{k}q^k e^{-nx}[nx]^k$$

we get

$$|f(x) - B_n^{p,q}(f;x)| = \sum_{k\in S_\delta}|f(x) - f(\frac{k}{n})|\binom{n}{k}q^k e^{-nx}[nx]^k +$$

$$\sum_{k\notin S_\delta}|f(x) - f(\frac{k}{n})|\binom{n}{k}q^k e^{-nx}[nx]^k \qquad (5)$$

We know $f \in C[0,1]$ and it is a bounded function, so $|f(x)| \leq M$ where $M > 0$ such that

$$|f(x) - f(\frac{k}{n})| \leq 2M, \forall x \in [0,1].$$

and hence

$$\Sigma_{k\in S_\delta}|f(x)-f(\frac{k}{n})|\binom{n}{k}q^k e^{-nx}[nx]^k$$
$$\leq 2M\Sigma_{k\in S_\delta}\binom{n}{k}q^k e^{-nx}[nx]^k$$

by equation (4)

$$\Sigma_{k\in S_\delta}|f(x)-f(\frac{k}{n})|\binom{n}{k}q^k e^{-nx}[nx]^k$$
$$\leq 2M\frac{1}{2n\delta^2} \tag{6}$$

Since function f is a continuous function and uniformly continuous also, then $\forall \in > 0$ then there exists $\delta > 0$ that depends on ϵ and f such that

$$|x-y|<\delta \Rightarrow |f(x)-f(y)|<\frac{\epsilon}{2}, \forall x, y \in [0,1]$$

then for $k \notin S_\delta$ so

$$\Sigma_{k\notin S_\delta}|f(x)-f(\frac{k}{n})|\binom{n}{k}q^k e^{-nx}[nx]^k$$
$$<\frac{\epsilon}{2}\sum_{k=0}^{n}\binom{n}{k}q^k e^{-nx}[nx]^k<\frac{\epsilon}{2} \tag{7}$$

by equation (6) and (7) we get

$$|f(x)-B_n^{p,q}(f;x)|<\frac{M}{2n\delta^2}+\frac{\epsilon}{2}$$

we choosing $N > \frac{M}{\in\delta^2}$ so

$$|f(x)-B_n^{p,q}(f;x)|<\epsilon, \forall n \geq N.$$

□

3.1. **Korovkin type theorem.**

Theorem 3.2. *If f is a function that is continuous on $[0,1]$ and $0<q<p\leq 1$, $\forall\ n \in N$, then $B_n^{p,q}(f,x) \to f(x)$ converges uniformly on $C[0,1]$.*

Proof. Since $[n+s] \to \infty$ when $s = 1,2,3$ as $n \to \infty$, then it is easily seen that $B_n^{p,q}(e_k;x) \to e^k$ *or* x^k where $k = 0,1,2$ and using the identity $[n+s]_{p,q} = S_{p,q}p^n + q^s[n]_{p,q}$ when s= 0,1,2. So we get our results due to the famous Korovkin's theorem. □

4. Rate of convergence

In this section we will study a rate of convergence. If $f \in C[0,1]$, then the modulus of continuity of function f such as

$$\omega(f,\delta) = \sup_{|t-x|<\delta,(x,t)\in[0,1]}|f(x)-f(t)|$$

and the Lipschitz maximal type functions of order λ as follows.

$$\widehat{\omega}_\lambda(f,\delta) = \lim_{t\neq x,(x,t)\in[0,1]}\frac{|f(t)-f(x)|}{|t-x|^\lambda}, 0<\lambda\leq 1$$

$$\omega_2(f,\delta) = \sup_{|h|\leq\delta}|f(x+2h)-2f(x+h)+f(x)| \quad where \quad x, x+h, x+2h \in [0,1]$$

Also, for a positive constant M, a Lipschitz function is one that is $f \in Lip_M(\phi)$ with $0 < \phi \leq 1$. Then

$$|f(t) - f(x)| \leq M|t-x|^{\phi}, \qquad \forall \quad t, x \in [0,1].$$

And Peetre K—functional as

$$K_2(f,\delta) = \inf_{g \in W^2} \{||f-g|| + \delta||g''||\}$$

where

$$W^2 = \{g \in C[0,1]; g', g'' \in C[0,1]\}$$

then existence of positive constant $C > 0$ such that

$$K_2(f,\delta) \leq C\omega_2(f, \sqrt{\delta}) \quad , d > 0.$$

Theorem 4.1. *For $f \in Lip_M(\phi)$ and $0 < q < p \leq 1$ and $n > 1$ then*

$$|B_n^{p,q}(f;x) - f(x)| \leq M\bigg(\Psi(n,p,q)\bigg)^{\phi}$$

where $\Psi = B_n^{p,q}\bigg(|t-x|;x\bigg)$.

Proof. For $f \in Lip_M(\phi)$ and $B_n^{p,q}(f;x)$ both are positive linear operators, then by Hölder's inequality, we get

$$|B_n^{p,q}(f;x) - f(x)| \leq B_n^{p,q}(|f(t)-f(x)|;x) \leq MB_n^{p,q}(|t-x|^{\phi};x)$$

If $\phi = 1$

$$|B_n^{p,q}(f;x) - f(x)| \leq MB_n^{p,q}(|t-x|^{\phi};x)$$
$$|B_n^{p,q}(f;x) - f(x)| \leq M\Psi(n,p,,q)^{\phi}$$

Hence proved. □

5. Direct estimates

Theorem 5.1. *([8]). If $f \in C[0,1]$ then*

$$|B_n^{p,q}(f(t) - f(x);x)| \leq 2\omega(f,\delta)$$

where

$$\lambda_n = \sqrt{B_n^{p,q}(t-x)^2;x}$$

Proof. By using Popoviciu's technique

$$|f(t) - f(x)| \leq \omega(f,\delta)\bigg(\frac{|t-x|}{\delta} + 1\bigg), \quad \forall \quad \delta > 0. \tag{8}$$

and also using linearity and monotonicity of the operator $D_n^{p,q}(f;x)$ we get

$$|B_n^{p,q}(f(t) - f(x));x| \leq B_n^{p,q}(|f(t)-f(x)|;x) \tag{9}$$

by (8) and (9) we have

$$|B_n^{p,q}(f(t) - f(x));x| \leq \omega(f,\delta)\bigg(\frac{B_n^{p,q}(|t-x|;x)}{\delta} + 1\bigg)$$

Now by the Cauchy-Schwartz inequality and the lemma for central moments, we have

$$|B_n^{p,q}(f(t)-f(x));x| \leq \omega(f,\delta)\left(\frac{B_n^{p,q}(|t-x|^2;x)^{1/2}}{\delta}+1\right)$$

$$\leq \omega(f,\delta)(\frac{\lambda_n}{\delta}+1)$$

if we take $\lambda_n = \delta$ then

$$\leq 2\omega(f,\delta) \quad So \quad proved.$$

□

Theorem 5.2. *If $f \in C[0,1]$ then we have*

$$|B_n^{p,q}(f;x)-f(x)| \leq C\omega^2(f;\delta_n^2(x)) + \omega(f,\frac{1}{n})$$

Proof. By lemma of central moments we get. Let

$$B_n^{p,q}\left((t-x)^2;x\right) \leq \delta_n(x)$$

and assume that

$$E_n^{p,q}(f;x) = f(x) - f\left(x + B_n^{p,q}(t-x;x)\right)$$

and

$$H_n^{p,q}(f;x) = E_n^{p,q} + B_n^{p,q}(f;x)$$

then we get

$$|E_n^{p,q}(f;x)| \leq \omega\left(f; B_n^{p,q}(t-x;x)\right) \leq \omega(f;\frac{1}{[n]}).$$

Now using Taylor's formula, we get

$$g(t) = g(x) + g'(x)(t-x) + \int_x^t (t-x)g''(u)du$$

so

$$H_n^{p,q}(g;x) - g(x) = g'(x)\left(H_n^{p,q}(t-x);x\right) + H_n^{p,q}\left(\int x^t(t-u)g''(u)du;x\right)$$

$$= B_n^{p,q}\left(\int x^t(t-u)g''(u)du;x\right) - \int_x^{x+B_n^{p,q}(t-x;x)}\left(x+B_n^{p,q}(t-x;x)-u\right)g''(u)du$$

we have

$$|H_n^{p,q}(g;x)-g(x)| \leq |B_n^{p,q}\left(\int x^t(t-u)g''(u)du;x\right)|$$

$$+|\int_x^{x+B_n^{p,q}(t-x;x)}\left(x+B_n^{p,q}(t-x;x)-u\right)g''(u)du|$$

$$\leq ||g''||B_n^{p,q}\left((t-x)^2;x\right) + \left(x+B_n^{p,q}(t-x;x)-u\right)^2||g''||$$

$$\leq \delta_n^2||g''||$$

also we get

$$|H_n^{p,q}(f;x)| \leq |B_n^{p,q}(f;x)| + 2||f|| \leq 3||f||$$

so

$$|B_n^{p,q} - f(x)| \leq |H_n^{p,q}(f-g;x) - (f-g)(x)| + |f\left(B_n^{p,q}(t-x);x\right) - f(x)| + |H_n^{p,q} - g(x)|$$

$$\leq |H_n^{p,q}(f-x;x)| + |(f-g)(x)|_{p,q} + |f\left(B_n^{p,q}(t-x);x\right) - f(x)| + |H_n^{p,q}(g;x) - g(x)|$$
$$\leq 4||f-g|| + \omega(f;\delta) + \delta_n^2(x)||g''||$$

taking infimum on RHS and we know $g \in W^2$ and using Peetre K- functional so

$$|B_n^{p,q}(f;x) - f(x)| \leq C\omega^2(f;\delta_n^2(x)) + \omega(f;\delta).$$

proved □

5.1. **Monotonicity for convex function.** In 2016 proved the monotonicity for (p,q)-Bernstein operators by Kang [16]. Now we shall study the monotonicity of the (p,q) Bernstein operators using the gamma function.

Definition 5.3. A function $f : R^n \to R$ is said to be convex if $\;forall\ \lambda \in [0,1]$ then

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$$

Example 5.4.

$$(1).f(x) = e^{ax}$$
$$(2).f(x) = Sin(\phi x)$$

.

The function $f(x) = e^{ax}$ exhibits convexity and monotonicity, as its values increase or decrease in correspondence with the behavior of $f(x)$. This property is similar to the second example, which depends on the value of $\phi = 1$ and is restricted to the interval $x \in (-\pi/2, \pi/2)$.

Theorem 5.5. *If $f \in C[0,1]$ is a convex function, then $B_n^{p,q}(f;x) \geq f(x)$, $\forall\ x \in [0,1]$, $\forall\ n \in N$ and $0 < q < p \leq 1$.*

Proof. We know that $f \in C[0,1]$ is a bounded function on $[0,1]$. and $|f| \leq M$ for $M > 0$ then we may write by using lemma $B_n^{p,q} > 0$ so we can

$$f(x) \leq B_n^{p,q}(f;x).$$

with alternating proof is that

$$let \quad x_k = [t] \quad and \quad \lambda_k = \binom{n}{k} q^k p^{-k} e^{-nx}[nx]^k$$

$$B_n^{p,q}(f;x) = [n]\sum_{k=0}^{n} \lambda_k \Gamma(k+1) f(x_k)$$
$$\geq f\left([n]\sum_{k=0}^{n} \lambda_k \Gamma(k+1)(x_k)\right)$$

so

$$B_n^{p,q}(f;x) \geq f(x).$$

Condition for Monotonicity on $[0,1]$: The generalized Bernstein operators with the polynomial $b_{n,k}(f;x)$ will be monotonically increasing on $[0,1]$ for $k \geq 1$. □

5.2. **Voronovskaja type theorem.** We present a significant quantitative Voronovskaja-type theorem in this section. Holhas also provided the identical first derivation theorem for

(p,q)-Bernstein operators [14], utilizing the smoothness modulus of Ditzian-Totik of the first order.

Theorem 5.6. *For any $f \in C[0,1]$,then the inequality holds:*

$$||B_n^{p,q} - f(x) - e^{-nx}\phi(x)f''(x)|| \leq Ce^{-nx}f''(x)\phi(x).$$

Proof. Since $f \in C[0,1]$ and $t, x \in [0,1]$. We know Taylor expansion,then we get:

$$f(t) - f(x) = (t-x)f'(x) + \int_x^t (t-u)f''(u)du$$

therefore

$$f(t)-f(x)-(t-x)f'(x) = \int_x^t (t-u)f''(u)du - \int_x^t (t-u)f''(x)du = \int_x^t (t-u)[f''(u)-f''(x)]du.$$

By using lemma 0.0.2 and 0.1.1 : we get

$$||B_n^{p,q} - f(x) - e^{-nx}\phi(x)f"(x)|| \leq B_n^{p,q}\left(\left|\int_x^t |t-u||f''(u) - f''(x)|du\right|; x\right) \qquad (A).$$

$\left|\int_x^t |t-u||f''(u) - f''(x)|du\right|$ was given by [11] page -337. as follows

$$\left|\int_x^t |t-u||f''(u) - f''(x)|du\right| \leq 2||f'' - g||(t-x)^2 = 2||\phi g'||||\phi^{-x}|||t-x|^3 \qquad (B).$$

Where $g \in W_\phi[0,1]$ $forall$ m = 1,2,3............... and $0 < q \leq p \leq 1$ there exist a constant $C_m > 0$.

$$||B_n^{p,q}((t-x)^m; x)|| \leq C_m\phi(x)e^{-nx} \qquad (C).$$

Where $x \in [0,1]$ and C is constant. Now combine (A), (B), and (C) with lemma 2.1
Then the Cauchy-Schwarz inequality we get is used by Kang[16].

$$||B_n^{p,q} - f(x) - e^{-nx}\phi(x)f''(x)|| \leq 2||f'' - g||B_n^{p,q}\left(|t-x|^2; x\right) + 2||\phi g'||||\phi^{-x}||B_n^{p,q}\left(|t-x|^3; x\right)$$

$$\leq 2||f'' - g||||\phi^x||e^{-nx} + 2||\phi g'||||\phi^{-x}||\left(B_n^{p,q}|t-x|^2; x\right)^{1/2}\left(B_n^{p,q}|t-x|^4; x\right)^{1/2}$$

$$\leq 2||f'' - g||||\phi^x||e^{-nx} + 2C||\phi g'||e^{-nx}\phi(x)$$

$$Ce^{-nx}\phi(x)\left(||f'' - g|| + ||\phi g''\right)$$

Since $\phi(x) \leq 3$ for every $x \in [0,1]$ we get

$$||B_n^{p,q} - f(x) - e^{-nx}\phi(x)f''(x)|| \leq 3C\phi(x)e^{-nx}\left(||f'' - g|| + ||\phi g''||\right)$$

Then finally we get

$$||B_n^{p,q} - f(x) - e^{-nx}\phi(x)f''(x)|| \leq Ce^{-nx}f''(x)\phi(x).$$

Hence Proved. □

6. Conclusions

This paper introduces a new modification of (p,q)-Bernstein operators. Using these operators, we propose and prove approximation properties for a new class of gamma functions in

(p, q) - calculus. We also studied Korvkin's theorem, direct estimates, and the rate of convergence through Peetre's K-functional and also proved the convergence theorem for Lipschitz functions of continuous functions. We provide a proof of Voronovskaja's theorem using the Ditzian-Totik modulus of smoothness and get flexible results for those theorems.

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