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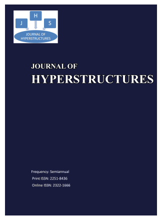


Research Paper

# SOME RESULTS ON DOMINATION IN ANNIHILATING-IDEAL GRAPHS OF COMMUTATIVE RINGS

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ABSTRACT

Let $R$ be a commutative ring with identity and let $\mathbb{A}(R)$ be the set of all ideals of $R$ with non-zero annihilators. The annihilating-ideal graph of $R$ is defined as the graph $\mathbb{AG}(R)$ with the vertex set $\mathbb{A}^*(R) = \mathbb{A}(R)\setminus\{(0)\}$ and two distinct vertices $I$ and $J$ are adjacent if and only if $IJ = (0)$. Let $G = (V, E)$ be a graph. A domination set for $G$ is a subset $S$ of $V$ such that every vertex not in $S$ is joined to at least one member of $S$ by some edge. The domination number $\gamma(G)$ is the minimum cardinality among the dominating sets of $G$. In this paper, we study and characterize the dominating sets and domination numbers of the annihilating-ideal graph $\mathbb{AG}(R)$ for a commutative ring $R$.

## 1. Introduction

The study of algebraic structures using the properties of graphs has been an exciting research topic in the last twenty years. There are many papers on assigning a graph to a ring, for instance see [3-18, 20]. Throughout this paper, all rings are assumed to be commutative with unity. For a ring $R$, we denote by $Z(R)$, $\text{Spec}(R)$, $\text{Min}(R)$ and $Ass(R)$, the set of all zero-divisors of $R$, the set of all prime ideals of $R$, the set of all minimal prime ideals of $R$ and the set of all associated prime ideals of $R$, respectively. A ring $R$ is said to be

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*reduced*, if it has no non-zero nilpotent elements or equivalently $\cap_{P\in \mathrm{Min}(R)}P = (0)$. A subset $S$ of a commutative ring $R$ is called a *multiplicative closed subset* ($m.c.s$) of $R$, if $1 \in S$ and $a, b \in S$ implies that $ab \in S$. If $S$ is an $m.c.s$ of $R$, then we denote by $R_S$, the ring of fractions of $R$. An ideal $I$ of $R$ is called *annihilating-ideal* if there exists a non-zero ideal $J$ of $R$ such that $IJ = (0)$. We use the notation $\mathbb{A}(R)$ for the set of annihilating-ideals of $R$. By the *annihilating-ideal graph* $\mathbb{AG}(R)$ of $R$, we mean the graph with vertices $\mathbb{A}^*(R) = \mathbb{A}(R)\diagdown\{(0)\}$ with two distinct vertices $I$ and $J$ adjacent if and only if $IJ = (0)$. Consequently, $\mathbb{AG}(R)$ is the empty graph if and only if $R$ is an *integral-domain*. The concept of the annihilating-ideal graph of a commutative ring was first introduced in [12]. Recently, set notion of the annihilating-ideal graph has been extensively studied by various authors (see for instance [1-6]).

Let $G = (V, E)$ be an undirected graph with vertex set $V$ and edge set $E$. We denotes the degree of a vertex $v$ in $G$ by $d(v)$. In addition, $N_G(v)$ called the *open neighborhood* of $v$ in $G$, denoted the set of vertices of $G$ which are adjacent to the vertex $v$ of $G$ and the *closed neighborhood* of $v$, $N_G[v] = N_G(v)\cup\{v\}$. Also, for any set $S \subseteq V(G)$, the *open neighborhood* of $S$, $N_G(S)$ is defined to be $\cup_{v\in S}N_G(v)$ and the *closed neighborhood* of $S$ is $N_G[S] = N_G(S)\cup S$. A set $S \subseteq V$ of vertices in a graph $G = (V, E)$ is called a *dominating set* if every vertex not in $S$ is joined to at least one member of $S$ by some edge, or equivalently, $N_G[S] = V(G)$. The minimum cardinality of a dominating set in $G$ is called the *domination number* of $G$ and is denoted by $\gamma(G)$. In addition, each dominating set of minimum cardinality is called a $\gamma$-set of $G$. Also, a *total dominating set* of a graph $G$ is a set $S$ of vertices of $G$ such that every vertex is adjacent to a vertex in $S$, or equivalently $N_G(S) = V$. The *total domination number* of $G$, denoted by $\gamma_t(G)$. We call a dominating set of cardinality $\gamma_t(G)$ a $\gamma_t$-set. A *semi-total dominating set* in $\mathbb{AG}(R)$ is a subset $S \subseteq \mathbb{A}^*(R)$ such that $S$ is a dominating set for $\mathbb{AG}(R)$ and for any $I \in S$ there is a vertex $J \in S$ (not necessarily distinct) such that $IJ = (0)$. The *semi-total* domination number $\gamma_{st}(\mathbb{AG}(R))$ of $\mathbb{AG}(R)$ is the minimum cardinality of a semi-total dominating set in $\mathbb{AG}(R)$. It is clear that for every ring $R$, $\gamma(\mathbb{AG}(R)) \leqslant \gamma_{st}(\mathbb{AG}(R)) \leqslant 2\gamma(\mathbb{AG}(R))$. A *clique* of a graph is a complete subgraph and the number of vertices in a largest clique of graph $G$, denoted by $\omega(G)$, is called the *clique number* of $G$. For a graph $G$, let $\chi(G)$ denote the *chromatic number* of $G$, i.e, the minimal number of colors which can be assigned to the vertices of $G$ in such a way that any two adjacent vertices have different colors. A dominating set $S$ is said to be a *clique dominating set*, if the induced subgraph $\langle S\rangle$ is a clique. The *clique domination number* $\gamma_{cl}(G)$ is the minimum cardinality of clique dominating set of $G$. Recall that graph $G$ is connected, if there is a path between every two distinct vertices. For distinct vertices $x$ and $y$ of $G$, let $d(x, y)$ be the length of the shortest path from $x$ to $y$ and if there is no such path we define $d(x, y) = \infty$. The *diameter* of $G$ is $\mathrm{diam}(G) = Sup\{d(x, y), x$ and $y$ are distinict vertices of $G\}$. A graph with $n$ vertices and no edge is denoted by $N_n$.

In [16], Nikanish and Maimani studied dominating sets of the annihilating-ideal graphs. The purpose of this paper is to general study on properties of dominating sets and domination numbers of the annihilating-ideal graphs of commutative rings. The organization of this paper is as follows:

In section 2, we discuss some basic properties and example of dominating sets of $\mathbb{AG}(R)$, for instants, we show that for each Artinian ring $R$, $\gamma_{st}(\mathbb{AG}(R)) \leqslant |\mathrm{Min}(R)|$ and hence $\gamma(\mathbb{AG}(R))$

is finite (see Proposition 2.5). Also, if $\gamma(\mathbb{AG}(R))$ is finite, then $Z(R) = \cup_{i=1}^{n}\mathrm{Ann}(I_i)$, where $I_i$'s are ideals of $R$ and the converse is also true if $\mathrm{Ann}(I_i) \in \mathrm{Spec}(R)$, for $1 \leqslant i \leqslant n$, consequently for every Noetherian ring $R$, $\gamma(\mathbb{AG}(R)) < \infty$ (see Proposition 2.9). In Theorem 2.16, it is shown that if $R$ is a Noetherian ring, then $\gamma_t(\mathbb{AG}(R)), \gamma_{st}(\mathbb{AG}(R)) \in \{1, 2, n\}$, where $n$ is number of maximal element in $Ass(R)$. Also, if $R$ is a ring, where $\mathrm{Max}(R)$ is a finite set and for each $\mathcal{M} \in \mathrm{Max}(R)$, $\gamma(\mathbb{AG}(R_{\mathcal{M}})) < \infty$, then $\gamma(\mathbb{AG}(R))$ is finite (see Theorem 2.12).

In section 3, we investigate domination numbers of the annihilating-ideal graph of ring $R$, where $R$ is a direct product of some rings. For instance, we show that, if $R$ is an Artinian ring such that $R \ncong F_1 \times F_2$, where $F_1, F_2$ are fields, then $\gamma(\mathbb{AG}(R)) = \gamma_{cl}(\mathbb{AG}(R)) = \gamma_{st}(\mathbb{AG}(R)) = n \leqslant \omega(\mathbb{AG}(R))$, where $n$ is number of summands in a decomposition of $R$ to local rings (see Proposition 3.3). In Proposition 3.5, it is shown that if $R$ is a ring which is not integral domain and $F$ is a field, then $\gamma(\mathbb{AG}(F \times R)) = \gamma_{st}(\mathbb{AG}(R)) + 1$. Finally, in Theorem 3.6, we show that, if $R = R_1 \times R_2$, where $R_1$, $R_2$ are two non-zero rings and $\gamma_{st}(\mathbb{AG}(R_1)) = m$ , $\gamma_{st}(\mathbb{AG}(R_2)) = n$, where $\mathbb{AG}(R_1)$ and $\mathbb{AG}(R_2)$ not empty. Then $\gamma(\mathbb{AG}(R)) \in \{1, 2, m+1, n+1, n+m\}$.

## 2. Some basic properties of dominating sets of $\mathbb{AG}(R)$

In this section we review some of the standard facts on domination numbers of the annihilating-ideal graphs. First we begin with the following example which is a direct result of [12] Proposition 1.3, Theorem 2.7 and Theorem 2.2, respectively.

*Example* 2.1.

(1) Let $(R, \mathcal{M})$ be an Artinian local ring. Then it is clear that for each $I \in \mathbb{A}^*(R)$, $(\mathrm{Ann}\mathcal{M})I = (0)$ and $(\mathrm{Ann}\mathcal{M})^2 = (0)$. Thus $\gamma(\mathbb{AG}(R)) = \gamma_{st}(\mathbb{AG}(R)) = \gamma_{cl}(\mathbb{AG}(R)) = 1$ and $\gamma_t(\mathbb{AG}(R)) \leqslant 2$.

(2) Let $R$ be a ring, where $Z(R)$ is an ideal of $R$ such that $(Z(R))^2 = (0)$, then $\gamma(\mathbb{AG}(R)) = \gamma_{st}(\mathbb{AG}(R)) = \gamma_{cl}(\mathbb{AG}(R)) = 1$ and $\gamma_t(\mathbb{AG}(R)) \leqslant 2$.

(3) Let $R$ be a ring. Then $\gamma(\mathbb{AG}(R)) = 1$ if and only if either $R = F \times D$, where $F$ is field and $D$ is an integral domain or $Z(R) = \mathrm{Ann}(x)$, for some $0 \neq x \in R$.

*Example* 2.2. The correctness of this example follows immediately from [13, Corollary 2.4] and [1, Theorem 2.3, corollary 11], respectively.

(1) Let $R$ be a ring such that $\mathbb{AG}(R) \cong K_{n,m}$, where $n, m \in \mathbb{N}$, then $\gamma(\mathbb{AG}(R)) = 1$.

(2) Let $R$ be a ring and $\mathbb{AG}(R)$ be a tree, then $\gamma(\mathbb{AG}(R)) \leqslant 2$ .

(3) Let $R$ be a ring such that $|\mathrm{Min}(R)| = 1$. If $\mathbb{AG}(R)$ is a bipartite graph, then $\gamma(\mathbb{AG}(R)) = 1$.

Let $R$ be a ring. The spectrum graph of $R$, denoted by $\mathbb{AG}_s(R)$, is the graph whose vertices are the set $\mathbb{A}_s(R) = \mathbb{A}^*(R) \cap \mathrm{Spec}(R)$ with distinct vertices $P$ and $Q$ adjacent if and only if $PQ = (0)$ (see [19]). The following propositions and theorems gives some properties of domination numbers of $\mathbb{AG}(R)$ via $\mathbb{AG}_s(R)$.

**Proposition 2.3.** *Let $R$ be a Noetherian ring. If $\mathbb{AG}_s(R)$ is a connected graph, then $\gamma(\mathbb{AG}(R)) \leqslant 2$.*

*Proof.* Since $\mathbb{AG}_s(R)$ is a connected graph, by [19, Theorem 3.7], $\mathbb{AG}_s(R) \cong K_1$, $K_2$ or $K_{1,\infty}$. If $\mathbb{AG}_s(R) \cong K_1$ or $K_{1,\infty}$, then by [19, Proposition 3.2], there exists a vertex of $\mathbb{AG}(R)$ which is adjacent to every other vertex of $\mathbb{AG}(R)$ and hence $\gamma(\mathbb{AG}(R)) = 1$. If $\mathbb{AG}_s(R) \cong K_2$, then by [19, Proposition 3.6], $\mathbb{AG}(R)$ is a complete bipartite graph and hence $\gamma(\mathbb{AG}(R)) \leqslant 2$. □

**Theorem 2.4.** *Assume that $R$ is a Noetherian ring such that $|\mathrm{Min}(R)| = 1$ and $\mathbb{AG}_s(R) \ncong N_2$. Then the following statements are equivalent.*

(1) *$\mathbb{AG}_s(R)$ is a connected graph.*
(2) *$\gamma(\mathbb{AG}(R)) = 1$.*
(3) *$Z(R) = \mathrm{Ann}x$, for some $0 \neq x \in R$.*

*Proof.* (1) ⇒ (2) Assume that $\mathbb{AG}_s(R)$ is a connected graph and $|\mathrm{Min}(R)| = 1$. By the same argument in previous proposition, if $\mathbb{AG}_s(R) \cong K_1$ or $K_{1,\infty}$, then $\gamma(\mathbb{AG}(R)) = 1$. If $\mathbb{AG}_s(R) \cong K_2$, then $|\mathrm{Min}(R)| = 1$ and [19, Proposition 3.6] implies that $\mathbb{AG}(R)$ is an star graph and $\gamma(\mathbb{AG}(R)) = 1$.

(2) ⇒ (3) Suppose that $\gamma(\mathbb{AG}(R)) = 1$, then by Example 2.1, either $R = F \times D$, where $F$ is field and $D$ is an integral domain or $Z(R) = \mathrm{Ann}x$, for some $0 \neq x \in R$, since $|\mathrm{Min}(R)| = 1$, we can conclude that $Z(R) = \mathrm{Ann}x$, where $0 \neq x \in R$.

(3) ⇒ (1) Assume that $Z(R) = \mathrm{Ann}x$, for some $0 \neq x \in R$, so $Rx$ is a vertex in $\mathbb{AG}(R)$, which is adjacent to every other vertex of $\mathbb{AG}(R)$. If $\mathbb{AG}_s(R) \cong K_2$, then there is nothing to proof. So we may assume that $|\mathbb{A}_s(R)| \neq 2$, thus by [19, Proposition 3.2], there is a vertex of $\mathbb{AG}_s(R)$ which is adjacent to every other vertex of $\mathbb{AG}_s(R)$. Therefore, $\mathbb{AG}_s(R)$ is a connected graph. □

**Theorem 2.5.** *Let $R$ be an Artinian ring. Then*

$$\gamma_{st}(\mathbb{AG}(R)) \leqslant |\mathrm{Min}(R)|.$$

*Proof.* Since $R$ is an Artinian ring. Then by [19, Theorem 3.10], $\mathbb{AG}_s(R) \cong K_1$, $\mathbb{AG}_s(R) \cong K_2$ or $\mathbb{AG}_s(R) \cong N_n$, where $n \geqslant 2$. Suppose that $\mathbb{AG}_s(R) \cong K_1$, thus $R$ is an Artinian local ring and hence $\gamma_{st}(\mathbb{AG}(R)) = 1 = |\mathrm{Min}(R)|$ (see Example 2.1 (1)). Now assume that $\mathbb{AG}_s(R) \cong K_2$, so $R \cong F_1 \times F_2$, where $F_1$, $F_2$ are fields, thus $\gamma_{st}(\mathbb{AG}(R)) = 2 = |\mathrm{Min}(R)|$. Finally assume that $\mathbb{AG}_s(R) \cong N_n$, where $n \geqslant 2$ and $V(\mathbb{AG}_s(R)) = \{P_1, \ldots, P_n\}$. In this case, $|\mathrm{Min}(R)| = n$. Since $\mathbb{AG}(R)$ is a connected graph (see [19, Theorem 2.1]) and $P_iP_j \neq (0)$ for $1 \leqslant i \neq j \leqslant n$, there exists ideal $I_i \in \mathbb{A}^*(R) \setminus \mathrm{Spec}(R)$ such that $I_iP_i = (0)$. For each $P_i$, select one $I_i$ and let $\mathbf{X} = \{I_i\}_{i=1}^n$. It is clear that $|\mathbf{X}| \leqslant n = |\mathrm{Min}(R)|$. We claim that $\mathbf{X}$ is a semi-total dominating set for $\mathbb{AG}(R)$. Assume that $J \in \mathbb{A}^*(R) \setminus (\mathbf{X} \cup V(\mathbb{AG}_s(R)))$, then by [19, Proposition 3.1], $J \subseteq P_i$, for some $1 \leqslant i \leqslant n$. Therefore $I_iJ = (0)$ and hence $\gamma(\mathbb{AG}(R)) \leqslant |\mathbf{X}| = n = |\mathrm{Min}(R)|$. Now assume that $I \in \mathbf{X}$, so there exists $1 \leqslant i \leqslant n$ such that $IP_i = (0)$. Let $1 \leqslant j \leqslant n$ and $i \neq j$, so $I \subseteq P_j$. On the other hand there exists $J \in \mathbf{X}$ such that $JP_j = (0)$, if $I = J$, then $I^2 = (0)$, otherwise $IJ = (0)$, therefore $\mathbf{X}$ is a semi-total dominating set for $\mathbb{AG}(R)$ and hence $\gamma_{st}(\mathbb{AG}(R)) \leqslant |\mathrm{Min}(R)| = n$. □

**Corollary 2.6.** *For every Artinian ring $R$, $\gamma(\mathbb{AG}(R))$ is finite.*

Let $R$ be an Artinian ring, the following proposition gives a relationship between chromatic number, clique number and diameter of $\mathbb{AG}(R)$, with $\gamma(\mathbb{AG}(R))$.

**Proposition 2.7.** *Let $R$ be an Artinian ring. Then*

(1) *If $\chi(\mathbb{AG}(R)) \leqslant 2$, then $\gamma(\mathbb{AG}(R)) = 1$.*

(2) *If* $\omega(\mathbb{AG}(R)) \leqslant 2$*, then* $\gamma(\mathbb{AG}(R)) = 1$.
(3) *If* $\mathrm{diam}(\mathbb{AG}(R)) \leqslant 2$*, then* $\gamma(\mathbb{AG}(R)) = 1$.

*Proof.* (1) Suppose that $\chi(\mathbb{AG}(R)) = 1$, since $\mathbb{AG}(R)$ is a connected graph (see [12, Theorem 2.1]), $\mathbb{AG}(R) \cong K_1$ and hence $\gamma(\mathbb{AG}(R)) = 1$. Now assume that $\chi(\mathbb{AG}(R)) = 2$ .By [13, Corollary 2.4], either $R \cong F_1 \times F_2$ or $R$ is a local ring. In every cases, $\gamma(\mathbb{AG}(R)) = 1$.
(2) If $\omega(\mathbb{AG}(R)) = 1$, then by [12, Theorem 2.1], $\mathbb{AG}(R) \cong K_1$ and hence $\gamma(\mathbb{AG}(R)) = 1$. Now assume that $\omega(\mathbb{AG}(R)) = 2$, so $\mathbb{AG}(R)$ is a triangle-free graph and hence [2, Corollary 2.5] implies that $\mathbb{AG}(R)$ is a bipartite graph. So by [13, Corollary 2.4], $\gamma(\mathbb{AG}(R)) = 1$.
(3) If $\mathrm{diam}(\mathbb{AG}(R)) = 0$ or $1$, then it is clear that $\gamma(\mathbb{AG}(R)) = 1$. Assume that $\mathrm{diam}(\mathbb{AG}(R)) = 2$. By [19, Theorem 4.2], $\mathbb{AG}_s(R) \cong K_1$ and hence $Z(R) = \mathrm{Ann}x$, where $0 \neq x \in R$ (see [19, Corollary 3.3]). Therefore $\gamma(\mathbb{AG}(R)) = 1$. □

The following example shows that the converse of Proposition 2.7 (1), (2) are not hold.

*Example* 2.8. Let $R = \dfrac{\mathbb{Z}_2[X]}{(X^5)}$. Then $R$ is an Artinian local ring with maximal ideal $\mathcal{M} = (X)$ and $\mathbb{A}^*(R) = \{(X), (X^2), (X^3), (X^4)\}$, therefore $\{(X^4)\}$ is a dominating set of $\mathbb{AG}(R)$ and hence $\gamma(\mathbb{AG}(R)) = 1$, but $\chi(\mathbb{AG}(R)) = 3 = \omega(\mathbb{AG}(R))$.

In the following results, we characterize when $\gamma(\mathbb{AG}(R))$ is finite.

**Proposition 2.9.** *Let* $R$ *be a ring. If* $\gamma(\mathbb{AG}(R))$ *is finite, then* $Z(R) = \cup_{i=1}^{n}\mathrm{Ann}(I_i)$*, where* $I_i$*'s are ideals of* $R$*. The converse is also true if* $\mathrm{Ann}(I_i) \in \mathrm{Spec}(R)$*, for* $1 \leqslant i \leqslant n$.

*Proof.* Suppose that $\gamma(\mathbb{AG}(R)) = m < \infty$ and $\mathbf{X} = \{J_1, \ldots, J_m\}$ be a dominating set of $\mathbb{AG}(R)$. Assume that $I \in \mathbb{A}^*(R)\setminus\mathbf{X}$, then, there is $1 \leqslant j \leqslant m$ such that $I \subseteq \mathrm{Ann}(J_i)$ and hence $Z(R) = \Big(\cup_{i=1}^{m} \mathrm{Ann}(J_i)\Big) \cup \Big(\cup_{i=1}^{m} J_i\Big)$. On the other hand $J_i \in \mathbb{A}^*(R)$ implies that $J_i \subseteq \mathrm{Ann}J$ for some ideal $J$ of $R$. Therefore $Z(R) = \cup_{i=1}^{n}\mathrm{Ann}(I_i)$, where $I_i$ is an ideal of $R$. Now assume that $Z(R) = \cup_{i=1}^{n}\mathrm{Ann}(I_i)$, where $I_i$'s are ideals of $R$ and $\mathrm{Ann}(I_i) \in \mathrm{Spec}(R)$, for $1 \leqslant i \leqslant n$. Let $\mathbf{X} = \{I_1, \ldots, I_n\}$, we claim that $\mathbf{X}$ is a dominating set for $\mathbb{AG}(R)$. Let $J \in \mathbb{A}^*(R)\setminus\mathbf{X}$. Since $J \subseteq Z(R) = \cup_{i=1}^{n}\mathrm{Ann}(I_i)$, by Prime Avoidance Theorem [18, Theorem 3.61], $J \subseteq \mathrm{Ann}(I_i)$ for some $1 \leqslant i \leqslant n$ and hence $JI_i = (0)$, so $\gamma(\mathbb{AG}(R)) < \infty$. □

**Corollary 2.10.** *For every Noetherian ring* $R$*,* $\gamma(\mathbb{AG}(R)) < \infty$.

*Proof.* Assume that $R$ is a Noetherian ring. By [18, Corollary 9.36], $Z(R) = \cup_{P \in Ass(R)}P$. Since $R$ is a Noetherian ring, $|Ass(R)| < \infty$ and hence $Z(R) = \cup_{i=1}^{n}\mathrm{Ann}(Rx_i)$, where $x_i \in R$ for $1 \leqslant i \leqslant n$. Therefore by Proposition 2.6, $\gamma(\mathbb{AG}(R)) < \infty$. □

The following theorem shows that if $R$ is a semilocal ring (i.e. $R$ has only finitely many maximal ideals) and for each maximal ideal $\mathcal{M}$ of $R$, $\gamma(\mathbb{AG}(R_{\mathcal{M}}))$ is finite, then $\gamma(\mathbb{AG}(R))$ is finite. First we need the following lemma.

**Lemma 2.11.** [2, Lemma 10] *Let* $R$ *be a ring and* $I$*,* $J$ *be two non-trivial ideals of* $R$*. If for each* $\mathcal{M} \in \mathrm{Max}(R)$*,* $I_{\mathcal{M}} = J_{\mathcal{M}}$*, then* $I = J$.

**Theorem 2.12.** *Let* $R$ *be a ring,* $\mathrm{Max}(R)$ *is a finite set and for each* $\mathcal{M} \in \mathrm{Max}(R)$*,* $\gamma(\mathbb{AG}(R_{\mathcal{M}})) < \infty$*, then* $\gamma(\mathbb{AG}(R)) < \infty$ .

*Proof.* Suppose that $\mathrm{Max}(R)$ is a finite set and $\mathrm{Max}(R) = \{\mathcal{M}_1, \dots, \mathcal{M}_n\}$. By contrary suppose that $\gamma(\mathbb{AG}(R)) = \infty$ and $\mathbf{X} = \{J_1, J_2, \dots\}$ is a infinite dominating set of $\mathbb{AG}(R)$. For ideal $\mathcal{M}_1$, let $\mathbf{X}_{\mathcal{M}_1} = \{(J_1)_{\mathcal{M}_1}, (J_2)_{\mathcal{M}_1}, \dots\}$. Assume that $I_{\mathcal{M}_1} \in \mathbb{A}^*(R_{\mathcal{M}_1})$, then $I \in \mathbb{A}^*(R)$ and there is $J_t \in \mathbf{X}$ such that $IJ_t = (0)$, so $I_{\mathcal{M}_1}(J_t)_{\mathcal{M}_1} = (0)$, and hence $\mathbf{X}_{\mathcal{M}_1}$ is a dominating set for $\mathbb{AG}(R_{\mathcal{M}_1})$, since $\gamma(\mathbb{AG}(R_{\mathcal{M}_1})) < \infty$, there exists infinite subset $\mathcal{A}_1 \subseteq \mathbb{N}$ such that for each $i, j \in \mathcal{A}_1$, $(J_i)_{\mathcal{M}_1} = (J_j)_{\mathcal{M}_1}$. Since $\gamma(\mathbb{AG}(R_{\mathcal{M}_2})) < \infty$, by same argument there exists $\mathcal{A}_2 \subseteq \mathbb{N}$ such that for every $i, j \in \mathcal{A}_2$, $(J_i)_{\mathcal{M}_2} = (J_j)_{\mathcal{M}_2}$. By continuing this procedure, there exists infinite subset $\mathcal{A} \subseteq \mathbb{N}$ such that for each $i, j \in \mathcal{A}$ and $\mathcal{M}_t$ for $1 \leqslant t \leqslant n$, $(J_i)_{\mathcal{M}_t} = (J_j)_{\mathcal{M}_t}$. Lemma 2.11 implies that $\mathbf{X}$ is a finite set, a contradiction and hence $\gamma(\mathbb{AG}(R)) < \infty$. □

In the next theorem, we characterize $\gamma_t(\mathbb{AG}(R))$ and $\gamma_{st}(\mathbb{AG}(R))$ for Noetherian ring $R$. First we need the following two lemmas.

**Lemma 2.13.** *Let $R$ be a ring such that $\gamma(\mathbb{AG}(R)) = 1$. Then*

$$\gamma_t(\mathbb{AG}(R)), \gamma_{st}(\mathbb{AG}(R)) \in \{1, 2\}.$$

*Proof.* Suppose that $\gamma(\mathbb{AG}(R)) = 1$, so there is a vertex $I \in \mathbb{A}^*(R)$ such that $I$ is adjacent to every other vertex of $\mathbb{AG}(R)$ and hence by [12, Theorem 2.2], either $R = F \times D$, where $F$ is a field and $D$ is an integral domain or $Z(R) = \mathrm{Ann}x$ for some $0 \neq x \in R$. If $Z(R) = \mathrm{Ann}x$, then $I = Rx$, implies that $x^2 = 0$ and hence $S = \{I\}$ is a $\gamma_{st}$-set for $\mathbb{AG}(R)$, so $\gamma_{st}(\mathbb{AG}(R)) = 1$ and $\gamma_t(\mathbb{AG}(R)) \leqslant 2$. Now assume that $R = F \times D$, in this case $J = F \times (0)$ is a vertex in $\mathbb{A}^*(R)$ which is adjacent to every other vertex of $\mathbb{AG}(R)$, where $J^2 \neq (0)$. Since $N(\{J\}) \cup \{J\} = \mathbb{A}^*(R)$, $\gamma_{st}(\mathbb{AG}(R)) = \gamma_t(\mathbb{AG}(R)) = 2$. □

**Corollary 2.14.** *For every local ring $R$, if $\gamma(\mathbb{AG}(R)) = 1$, then $\gamma_{st}(\mathbb{AG}(R)) = 1$.*

*Proof.* It is clear with Lemma 2.13. □

**Lemma 2.15.** [10, Lemma 3.6] *Let $x$ and $y$ be elements in $R$ such that $\mathrm{Ann}(Rx)$ and $\mathrm{Ann}(Ry)$ are two distinct prime ideals of $R$. Then $xy = 0$.*

**Theorem 2.16.** *Let $R$ be a Noetherian ring. Then*

$$\gamma_t(\mathbb{AG}(R)), \gamma_{st}(\mathbb{AG}(R)) \in \{1, 2, n\}$$

*where $n$ is number of maximal element in $Ass(R)$.*

*Proof.* If $\gamma(\mathbb{AG}(R)) = 1$, then by Lemma 2.13, we have done. Then we assume that $\gamma(\mathbb{AG}(R)) \neq 1$ and $\mathbf{X} = \{P_1, \dots, P_n\}$ is the set of maximal element of $Ass(R)$. By [18, Corollary 9.36], $Z(R) = \cup_{i=1}^n P_i$, where $P_i = \mathrm{Ann}(Rx_i)$. Let $\overline{\mathbf{X}} = \{Rx_i\}_{i=1}^n$. We claim that $\overline{\mathbf{X}}$ is a $\gamma_t$-set and a $\gamma_{st}$-set for $\mathbb{AG}(R)$. Suppose that $I \in \mathbb{A}^*(R)$, by Prime Avoidance Theorem, for some $1 \leqslant i \leqslant n$, $I \subseteq \mathrm{Ann}(Rx_i)$ and hence $I(Rx_i) = (0)$. By Lemma 2.10 for each $1 \leqslant i, j \leqslant n$, $(Rx_i)(Rx_j) = (0)$ and hence $\overline{\mathbf{X}}$ is a semi-total dominating set of $\mathbb{AG}(R)$. Now assume that $\gamma_{st}(\mathbb{AG}(R)) = m$. It is clear that $m \leqslant n$ and there exists $\mathbf{Y} = \{I_1, I_2 \dots, I_m\} \subseteq \mathbb{A}^*(R)$ such that for each $J \in \mathbb{A}^*(R) \diagdown \mathbf{Y}$, $JI_i = (0)$, for some $1 \leqslant i \leqslant m$, so $J \subseteq \mathrm{Ann}(I_i)$. Also for each $I_i \in \mathbf{Y}$, $I_i \subseteq \mathrm{Ann}(I_j)$, for some $1 \leqslant j \leqslant n$, thus $\cup_{i=1}^n P_i = Z(R) = \cup_{j=1}^m \mathrm{Ann}(I_j)$. By Prime Avoidance Theorem, for each $1 \leqslant j \leqslant m$, there is $1 \leqslant i \leqslant n$ such that $\mathrm{Ann}(I_j) \subseteq P_i$, therefore $Z(R) = \cup_{j=1}^m P_j$. Now assume that $K \in \mathbf{X}$, then for some $1 \leqslant j \leqslant m$, $K \subseteq P_j$. Since $K$ is maximal in $Ass(R)$, so $K = P_j$ and hence $n = |\mathbf{X}| \leqslant |\mathbf{Y}| = m$, therefore $\gamma_{st}(\mathbb{AG}(R)) = \gamma_t(\mathbb{AG}(R)) = n$. □

We conclude this section with the following proposition.

**Proposition 2.17.** *Let $R$ be a ring and $S$ be an m.c.s of ring $R$ containing no zero-divisors. Then $\gamma_{cl}(\mathbb{AG}(R_S)) \leqslant \gamma_{cl}(\mathbb{AG}(R))$. Moreover $\gamma_{cl}(\mathbb{AG}(R_S)) = \gamma_{cl}(\mathbb{AG}(R))$, when $R$ is a reduced ring.*

*Proof.* Since for each $I_S, J_S \in \mathbb{A}^*(R_S)$, where $I_S \neq J_S$ and $I_S J_S = (0)$, we have $I \neq J$ and $IJ = (0)$, so we can conclude that $\gamma_{cl}(\mathbb{AG}(R_S)) \geqslant \gamma_{cl}(\mathbb{AG}(R))$. Now assume that $R$ is a reduced ring. We claim that for each $I, J \in \mathbb{A}^*(R)$ with $I \neq J$ and $IJ = (0)$, $I_S \neq J_S$ and $I_S J_S = (0)$. By contrary suppose that for some $I, J \in \mathbb{A}^*(R)$ such that $I \neq J$, we have $I_S = J_S$. Therefore $I_S^2 = I_S I_S = I_S J_S = (IJ)_S = (0)$ and hence $I_S = (0)$ a contradiction. So $\gamma_{cl}(\mathbb{AG}(R_S)) \leqslant \gamma_{cl}(\mathbb{AG}(R))$ and hence equality is hold. □

## 3. Dominating numbers of the annihilating-ideal graph of a direct product of rings

In this section we investigate domination numbers of ring $R$, where $R$ is a direct product of rings. We begin with the following proposition.

**Proposition 3.1.** *Let $R$ be a ring such that $R = R_1 \times R_2$, where $R_1$ and $R_2$ are not integral domain. Then $\gamma_{st}(\mathbb{AG}(R)) \leqslant \gamma_{st}(\mathbb{AG}(R_1)) + \gamma_{st}(\mathbb{AG}(R_2))$.*

*Proof.* Let $\gamma_{st}(\mathbb{AG}(R_1)) = \infty$ or $\gamma_{st}(\mathbb{AG}(R_2)) = \infty$, then there is nothing to proof. Assume that $\gamma_{st}(\mathbb{AG}(R_1)) = m$ and $\gamma_{st}(\mathbb{AG}(R_2)) = n$, where $\mathbf{A} = \{I_1, \ldots, I_m\}$ and $\mathbf{B} = \{J_1, \ldots, J_n\}$ are $\gamma_{st}$-set for $\mathbb{AG}(R_1)$ and $\mathbb{AG}(R_2)$, respectively. Let $\mathbf{A}_1 = \{I \times (0);\ I \in \mathbf{A}\}$ and $\mathbf{B}_1 = \{(0) \times J;\ J \in \mathbf{B}\}$. We claim that $\mathbf{X} = \mathbf{A}_1 \cup \mathbf{A}_2$ is a semi-total dominating set for $R$. Assume that $K \times L \in \mathbb{A}^*(R) \setminus \mathbf{X}$. If either $K = (0)$ or $L = (0)$, then it is clear that $K \times L$ is adjacent to a vertex in $\mathbf{X}$. We may assume that $K, L \neq (0)$. Suppose that $K = R_1$, since $L \in \mathbb{A}^*(R_2)$ for some $1 \leqslant t \leqslant n$, there exists $J_t \in \mathbf{B}$ such that $LJ_t = (0)$. This implise that $(R_1 \times L)((0) \times J_t) = (0) \times (0)$ and hence $R_1 \times L$ is adjacent to a vertex in $\mathbf{X}$. For case $L = R_2$ we have a similar argument. Now assume that $K \neq (0)$, $R_1$ and $L \neq (0)$, $R_2$. Since $K \in \mathbb{A}^*(R_1)$ for some $J_t \in \mathbf{B}$, where $1 \leqslant t \leqslant n$, $LJ_t = (0)$ and $(K \times L)((0) \times J_t) = (0) \times (0)$. On the other hand it is clear that every vertex in $\mathbf{X}$ is adjacent to a vertex in $\mathbf{X}$, so $\gamma_{st}(\mathbb{AG}(R)) \leqslant |\mathbf{X}| = m + n$. □

The following example shows that the converse of the Proposition 3.1 is not hold.

*Example* 3.2. Let $R_1 = \mathbb{Z}_4$, $R_2 = \mathbb{Z}_6$ and $R = R_1 \times R_2$. It is clear that $\mathbf{A} = \{(\bar{2})\}$ and $\mathbf{B} = \{(\bar{2}), (\bar{3})\}$ are $\gamma_{st}$-set for $\mathbb{AG}(R_1)$ and $\mathbb{AG}(R_2)$, respectively. Also $\mathbf{X} = \{(\bar{2}) \times (0), (0) \times (\bar{3})\}$ is a $\gamma_{st}$-set for $\mathbb{AG}(R)$. Therefore $\gamma_{st}(\mathbb{AG}(R_1)) = 1$, $\gamma_{st}(\mathbb{AG}(R_2)) = 2$ and $\gamma_{st}(\mathbb{AG}(R)) = 2$.

**Proposition 3.3.** *Let $R$ be an Artinian ring such that $R \not\cong F_1 \times F_2$, where $F_1$, $F_2$ are fields. Then*

$$\gamma(\mathbb{AG}(R)) = \gamma_{cl}(\mathbb{AG}(R)) = \gamma_{st}(\mathbb{AG}(R)) = n \leqslant \omega(\mathbb{AG}(R))$$

*, where $n$ is number of summands in a decomposition of $R$ to local rings.*

*Proof.* First assume that $R$ is a local ring. Since $R$ is an Artinian ring, Example 2.1 (1) implies that $\gamma(\mathbb{AG}(R)) = \gamma_{cl}(\mathbb{AG}(R)) = \gamma_{st}(\mathbb{AG}(R)) = 1 = n \leqslant \omega(\mathbb{AG}(R))$. Now assume that $R$ is an Artinian ring which is not local, by [9, Theorem 8.7], $R \cong R_1 \times \cdots \times R_n$, where $n \geqslant 2$ and $R_i$'s are Artinian local ring with maximal ideal $\mathcal{M}_i$, for $1 \leqslant i \leqslant n$. It is sufficient to proof for case $n = 2$ (for $n \geqslant 3$, we have a similar argument). Let $R \cong R_1 \times R_2$, then $\mathrm{Max}(R) = \{\mathcal{M}_1 \times R_2, R_1 \times \mathcal{M}_2\}$. Since $R$ is an Artinian ring by [12, Proposition 1.3], $(\mathcal{M}_1 \times R_2), (R_1 \times \mathcal{M}_2) \in \mathbb{A}^*(R)$, where $(\mathcal{M}_1 \times R_2)(R_1 \times \mathcal{M}_2) \neq (0) \times (0)$. It is clear that there is nothing non-zero ideal of $R$ which is adjacent to $\mathcal{M}_1 \times R_2$ and $R_1 \times \mathcal{M}_2$. Since $\mathbb{AG}(R)$ is a connected graph (see [12, Theorem 2.1]), there are two ideals $I_1 \times (0)$ and $(0) \times J_1$ such that $I_1 \times (0) \subseteq \mathrm{Ann}(\mathcal{M}_1 \times R_2)$ and $(0) \times J_1 \subseteq \mathrm{Ann}(R_1 \times \mathcal{M}_2)$. Now assume that $I \times J \in \mathbb{A}^*(R)$, then $I \times J \subseteq (\mathcal{M}_1 \times R_2) \cap (R_1 \times \mathcal{M}_2)$, so $I \times J \subseteq \mathrm{Ann}(I_1 \times (0)) \cap \mathrm{Ann}((0) \times J_1)$ and hence $\mathbf{X} = \{I_1 \times (0), (0) \times J_1\}$ is a dominating set of $\mathbb{AG}(R)$, then $\gamma(\mathbb{AG}(R)) \leqslant 2$. Since there is no any vertex of $\mathbb{AG}(R)$ which is adjacent to every other vertex of $\mathbb{AG}(R)$, then $\gamma(\mathbb{AG}(R)) = 2$. Now Assume that $G$ is a subgraph of $\mathbb{AG}(R)$, such that $\mathrm{V}(G) = \mathbf{X}$. Since $G$ is a complete graph, $\mathbf{X}$ is a clique dominating set and hence $\gamma(\mathbb{AG}(R)) = \gamma_{cl}(\mathbb{AG}(R)) = \gamma_{st}(\mathbb{AG}(R)) = 2 = n \leqslant \omega(\mathbb{AG}(R))$. □

**Corollary 3.4.** *Let $R$ be a non-domain Artinian reduced ring. Then*

(1) *If $R \cong F_1 \times F_2$, where $F_1$, $F_2$ are fields, then $\gamma_{st}(\mathbb{AG}(R)) = \omega(\mathbb{AG}(R)) = 2$.*
(2) *If $R \ncong F_1 \times F_2$, where $F_1$, $F_2$ are fields, then*

$$\gamma(\mathbb{AG}(R)) = \gamma_{st}(\mathbb{AG}(R)) = \gamma_{cl}(\mathbb{AG}(R)) = \omega(\mathbb{AG}(R))$$

*Proof.* (1) It is clear.
(2) Since $R$ is an Artinian reduced ring, it is well known that $R \cong F_1 \times \cdots \times F_n$, where $F_i$'s are fields and $n \geqslant 3$. Let $\mathbf{X} = \{F_1 \times (0) \times \cdots \times (0), \ldots, (0) \times \cdots \times F_n\}$. It is clear that $\mathbf{X}$ is a $\gamma$-set and maximal clique for $\mathbb{AG}(R)$ and hence $\gamma(\mathbb{AG}(R)) = \gamma_{st}(\mathbb{AG}(R)) = \gamma_{cl}(\mathbb{AG}(R)) = \omega(\mathbb{AG}(R)) = n$. □

**Proposition 3.5.** *Let $R$ be a ring which is not integral domain and $F$ be a field. Then*

$$\gamma(\mathbb{AG}(F \times R)) = \gamma_{st}(\mathbb{AG}(R)) + 1$$

*Proof.* Assume that $\gamma_{st}(\mathbb{AG}(R)) = n$ and $\mathbf{X} = \{I_1, \ldots, I_n\}$ is a $\gamma_{st}$-set for $\mathbb{AG}(R)$. It is clear that $\mathbf{Y} = \{(0) \times I_i,\ I_i \in \mathbf{X}\} \cup \{F \times (0)\}$ is a dominating set for $\mathbb{AG}(F \times R)$. Since $|\mathbf{Y}| = n+1$, $\gamma(\mathbb{AG}(F \times R)) \leqslant n+1$. Now assume that $\mathbf{A}$ is a $\gamma$-set for $\mathbb{AG}(F \times R)$. Let $\mathbf{B} = \{I, (0) \times I \in \mathbf{A}\}$. We claim that $\mathbf{B}$ is a semi-total dominating set for $\mathbb{AG}(R)$. Assume that $J \in \mathbb{A}^*(R) \setminus \mathbf{B}$, then $F \times J \in \mathbb{A}^*(F \times R)$, so there exists $I_1 \times J_1 \in \mathbf{A}$ such that $I_1 \times J_1 \subseteq \mathrm{Ann}(F \times J)$ and hence $I_1 = J_1 J = (0)$, thus $J_1 \in \mathbf{B}$. Therefore $\mathbf{B}$ is a dominating set for $\mathbb{AG}(R)$. Now suppose that $I \in \mathbf{B}$, so $F \times I \in \mathbb{A}^*(F \times R)$, thus there is $I_1 \times J_1 \in \mathbf{A}$ such that $I_1 \times J_1 \subseteq \mathrm{Ann}(F \times I)$, thus $I_1 = IJ_1 = (0)$ and $J_1 \in \mathbf{B}$ and $IJ_1 = (0)$. Therefore $\mathbf{B}$ is a semi-total dominating set for $\mathbb{AG}(R)$. Now we claim that $|\mathbf{B}| < |\mathbf{A}|$. By contrary suppose that $|\mathbf{A}| = |\mathbf{B}|$. It is clear that $(0) \times R \in \mathbb{A}^*(F \times R)$, but for each $I \in \mathbf{B}$, $((0) \times R)((0) \times I) \neq (0) \times (0)$, a contradiction. Therefore $|\mathbf{A}| = \gamma(\mathbb{AG}(F \times R)) \geqslant |\mathbf{B}| + 1 \geqslant \gamma_{st}(\mathbb{AG}(R)) + 1$ and hence $\gamma(\mathbb{AG}(F \times R)) = \gamma_{st}(\mathbb{AG}(R)) + 1$. □

We conclude this paper with the following theorem.

**Theorem 3.6.** *Let $R = R_1 \times R_2$, where $R_1$ and $R_2$ are two non-zero rings such that $\gamma_{st}(\mathbb{AG}(R_1)) = m$ , $\gamma_{st}(\mathbb{AG}(R_2)) = n$. Then*

$$\gamma(\mathbb{AG}(R)) \in \{1, 2, m+1, n+1, n+m\}$$

*Proof.* We consider all cases for $\mathbb{A}^*(R_1)$ and $\mathbb{A}^*(R_2)$. First assume that $\mathbb{A}^*(R_1) = \mathbb{A}^*(R_2) = \emptyset$ and $I \times J \in \mathbb{A}^*(R)$. It is clear that either, $I = (0)$ or $J = (0)$ and hence $\mathbf{X} = \{R_1 \times (0), (0) \times R_2\}$ is a dominating set for $\mathbb{AG}(R)$and thus $\gamma(\mathbb{AG}(R)) \leqslant 2$. Now assume that $\mathbb{A}^*(R_1) \neq \emptyset$ and $\mathbb{A}^*(R_2) = \emptyset$. In this case, $\mathbb{A}^*(R) = \{I \times J,\ I \in \mathbb{A}^*(R_1), J \text{ is an ideal of } R_2\} \cup \{(0) \times J : J$ is an non-zero ideal of $R\} \cup \{I \times (0),\ \text{where } (0) \neq I \notin \mathbb{A}^*(R_1)\}$. Suppose that $\mathbf{A}$ is a $\gamma_{st}$-set for $R_1$ and $\mathbf{B} = \{I \times (0) :\ I \in \mathbf{A}\} \cup \{(0) \times R_2\}$. It is clear that $\mathbf{B}$ is a dominating set (also a semi-total dominating set) for $\mathbb{AG}(R)$ and hence $\gamma(\mathbb{AG}(R)) \leqslant |\mathbf{A}| + 1 = m + 1$. Let $\mathbf{C}$ be a $\gamma$-set for $\mathbb{AG}(R)$ and $\mathbf{D} = \{I,\ I \times (0) \in \mathbf{C}\}$. We claim that $\mathbf{D}$ is a semi-total dominating set for $\mathbb{AG}(R_1)$. Assume that $I \in \mathbb{A}^*(R_1)$, then $I \times R_2 \in \mathbb{A}^*(R)$. Since $\mathbf{C}$ is a semi-total dominating set for $\mathbb{AG}(R)$, there exists $I_1 \times J_1 \in \mathbf{C}$ such that $I_1 \times J_1 \subseteq \mathrm{Ann}(I \times R_2)$, so $J_1 = (0)$ and $II_1 = (0)$, then $I_1 \in \mathbf{D}$ and hence $\mathbf{D}$ is a dominating set for $\mathbb{AG}(R)$. Let $I \in \mathbf{D}$, so $I \times R_2 \in \mathbb{A}^*(R)$ and hence there exists $L \times K \in \mathbf{C}$ such that $I \times R_2 \subseteq \mathrm{Ann}(L \times K)$, so $K = (0)$ and $LI = (0)$, thus $L \in \mathbf{D}$ and hence $\mathbf{D}$ is a semi-total dominating set for $\mathbb{AG}(R_1)$, therefore $\gamma_{st}(\mathbb{AG}(R_1)) \leqslant |\mathbf{D}| \leqslant |\mathbf{C}| = \gamma(\mathbb{AG}(R))$. If $|\mathbf{C}| = \gamma_{st}(\mathbb{AG}(R_1))$, then $\mathbf{C} = \{I \times (0) :\ I \in \mathbf{D}\}$ and $R_1 \times (0)$ is a vertex of $\mathbb{AG}(R)$ such that for each $L \times K \in \mathbf{C}$, $(R_1 \times (0))(L \times K) \neq (0) \times (0)$, a contradiction, so $\gamma(\mathbb{AG}(R)) \geqslant \gamma_{st}(\mathbb{AG}(R_1)) + 1$ and hence $\gamma(\mathbb{AG}(R)) = m + 1$. For case $\mathbb{A}^*(R_1) = \emptyset$ and $\mathbb{A}^*(R_2) \neq \emptyset$, by same argument we have $\gamma(\mathbb{AG}(R)) = n + 1$. Finally assume that $\mathbb{A}^*(R_1) \neq \emptyset$ and $\mathbb{A}^*(R_2) \neq \emptyset$. Suppose that $\mathbf{A}$ is a $\gamma$-set for $\mathbb{AG}(R)$ and $\mathbf{B} = \{I,\ I \times (0) \in \mathbf{A}\}$ and $\mathbf{C} = \{J,\ (0) \times J \in \mathbf{A}\}$. By same argument in before case, $\mathbf{B}$ is a semi total dominating set for $\mathbb{AG}(R_1)$ and $\mathbf{C}$ is a semi-total dominating set for $\mathbb{AG}(R_2)$, consequently $m = \gamma_{st}(\mathbb{AG}(R_1)) \leqslant |\mathbf{B}|$ and $n = \gamma_{st}(\mathbb{AG}(R_2)) \leqslant |\mathbf{C}|$. Therefore $\gamma(\mathbb{AG}(R)) = |\mathbf{A}| \geqslant |\mathbf{B}| + |\mathbf{C}| \geqslant m + n$. On the other hand by Proposition 3.1, $\gamma(\mathbb{AG}(R)) \leqslant \gamma_{st}(\mathbb{AG}(R)) \leqslant m + n$. Thus $\gamma(\mathbb{AG}(R)) = m + n$. Therefore in general, $\gamma(\mathbb{AG}(R)) \in \{1, 2, m+1, n+1, n+m\}$. □

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