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Research Paper

CONNECTING GRAPHS WITH R-HYPERMODULES VIA NORMAL FUZZY SUBHYPERMODULES

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ABSTRACT

In this paper, we analyze the connection between R-hypermodules and graphs by associating a graph with an R-hypermodule through a normal fuzzy subhypermodule. We investigate the graph's properties, including connectedness, completeness, Eulerian and Hamiltonian characteristics. By defining a regular relation based on the fuzzy subhypermodule, we study how algebraic properties of R-hypermodules influence the associated graph. This work contributes to the understanding of fuzzy algebraic structures and their graphical representations, with potential applications in computer science and network theory.

1. Introduction

In 1934, French mathematician Marty introduced the concept of hyperoperations at the 8th Mathematical Congress of the Scandinavian countries, laying the foundation for hypergroups as a generalization of group theory ([1]). This marked the beginning of the theory of algebraic hyperstructures. Concurrently, Zadeh's introduction of the fuzzy set theory in 1965 ([2]) opened new avenues for exploring the intersection of fuzzy logic and various branches of mathematics and engineering. In 1971, Rosenfeld further advanced this by introducing fuzzy

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algebraic structures, specifically the fuzzy subgroupoid of a groupoid ([3]). Since then, researchers have expanded the field, connecting fuzzy theory with algebraic hyperstructures, giving rise to fuzzy hyperstructures.

Graph theory has been widely applied across various scientific disciplines, such as computer science, image processing, and networking. Recently, graph representations have been used to study algebraic structures like rings and groups, revealing insights into their properties ([4], [5], [6]). Finally, the study of algebraic hyperstructures through the lens of graph theory has become an intriguing area of research. One notable example is the exploration of the relationship between hypergroups and hypergraphs, as discussed by Corsini ([7]). For further reading on the relationship between the theory of algebraic huperstructures and graph theory, you may refer to ([8], [9], [10], [11], [12], [13]).

In this paper, we examine the relationship between graph theory and algebraic hyperstructures, specifically focusing on R-hypermodules. By considering normal fuzzy subhypermodules of R-hypermodules and the regular relations they induce, we associate a graph to the R-hypermodule and investigate its structural properties, contributing to the study of fuzzy hyperstructures. We also highlight the concept of normal fuzzy subhypermodules, as introduced by Zhan, Davvaz, and Shum ([14]).

2. Preliminaries

In this section, we recall several key definitions introduced by earlier researchers in the field.

Let H be a non-empty set with a mapping $\circ : H \times H \longrightarrow P^{\star}(H)$, where $P^{\star}(H)$ denotes the set of all non-empty subsets of H. The algebraic structure $(H, \circ)$ is referred to as a hypergroupoid. For sets $A, B \in P^{\star}(H)$, the hyperoperation $A \circ B$ is defined as $A \circ B = \bigcup\{a \circ b : a \in A, b \in B\}$. Similarly, for an element $x \in H$, we use $x \circ A$ and $A \circ x$ to represent $\{x\} \circ A$ and $A \circ \{x\}$, respectively.

A hypergroupoid $(H, \circ)$ is called semihypergroup if, for each $x, y, z \in H$, the following holds:

$$x \circ (y \circ z) = (x \circ y) \circ z$$

If the hyperoperation $\circ$ is commutative, i.e., $x \circ y = y \circ x$ for all $x, y \in H$, then $(H, \circ)$ is referred to as a commutative semihypergroup. A commutative semihypergroup $(H, \circ)$ is called a canonical hypergroup if it satisfies the following axioms:

(i) There exists a unique $0 \in H$ such that $0 \circ x = x$, for all $x \in H$;
(ii) For each $x \in H$, there exists a unique element $x' \in H$ such that $0 \in x \circ x'$. (we call x' the opposite of x).
(iii) For all $x, y, z \in H$, if $x \in y \circ z$, then $z \in y' \circ x$ and $y \in x \circ z'$.

Next, we recall the concepts of Krasner hyperrings and hypermodules, which were formally defined in earlier works ([15], [14]).

Definition 2.1. A Krasner hyperring is an algebraic system $(R, +, \cdot)$ which satisfies the following axioms:

(i) $(R, +)$ is a canonical hypergroup (where we use $-x$ to denote the opposite of x);
(ii) $(R, \cdot)$ is a semigroup having zero as a bilaterally absorbing element;
(iii) The multiplication operation "." is distributive over the hyperoperation "+".

A Krasner hyperring R has a unit element if there exists an element $1 \in R$ such that for all $r \in R$, $r \cdot 1 = 1 \cdot r = r$.

Definition 2.2. A canonical hypergroup $(M, +)$ is called a left hypermodule over a hyperring R if there exists a map $\cdot : R \times M \longrightarrow P^{\star}(M)$ such that for all $r_1, r_2 \in R$ and $m_1, m_2 \in M$, the following conditions hold:

(i) $r_1 \cdot (m_1 + m_2) = r_1 \cdot m_1 + r_1 \cdot m_2$;
(ii) $(r_1 + r_2) \cdot m_1 = r_1 \cdot m_1 + r_2 \cdot m_1$;
(iii) $(r_1 r_2) \cdot m_1 = r_1 \cdot (r_2 \cdot m_1)$.

A hypermodule M over a Krasner hyperring R is unitary if for all $m \in M$, $m \in 1 \cdot m \cap m \cdot 1$.

Throughout this paper, R is a Krasner hyperring and M is a left hypermodule over R. When we refer to an "R-hypermodule," we mean a left R-hypermodule. A non-empty subset A of an R-hypermodule $(M, +)$ is called a subhypermodule if $(A, +)$ itself forms an R-hypermodule. Furthermore A is a normal subhypermodule if $x + A - x \subseteq A$ for all $x \in A$.

Definition 2.3. Let M and M' be two R-hypermodules. A map $f : M \to M'$ is called an R-homomorphism if it satisfies the following conditions for all $a, b \in M$ and $r \in R$:

- $f(a + b) = f(a) + f(b)$,
- $f(r \cdot a) = r \cdot f(a)$
- $f(0) = 0$,

where

$$f(a + b) = \cup\{f(z) : z \in a + b)\} \text{ and } f(r \cdot a) = \cup\{f(z) : z \in r \cdot a\}.$$

Furthermore, f is an isomorphism if f is both injective and surjective. We denote an isomorphism between M and M' as $M \cong M'$.

Fuzzy Subhypermodules

Definition 2.4. [2] Let X be a set. A fuzzy subset of X, is a function $\mu : X \to [0, 1]$. For fuzzy subsets μ and ν of X, we say that μ is contained in ν, denoted $\mu \subseteq \nu$, if for all $x \in X$, $\mu(x) \leq \nu(x)$.

Definition 2.5. [16] Let X and X' be sets, and let $f : X \to X'$ be a function. Let μ and ν be fuzzy subsets of X and X', respectively. The image $f(\mu)$ of μ is a fuzzy subset of X' defined by:

$$f(\mu)(y) = \begin{cases} \bigvee\limits_{x \in f^{-1}(y)} \mu(x) & if \;\; f^{-1}(y) \neq \emptyset \\ 0 & otherwise \end{cases}$$

for all $y \in X'$. The inverse image $f^{-1}(\nu)$ of ν is a fuzzy subset of X defined by $f^{-1}(\nu)(x) = \nu(f(x))$ for all $x \in X$.

In the following, we recall several key concepts related to fuzzy subhypermodules, as defined in ([14]).

Definition 2.6. A fuzzy subset μ of an R-hypermodule M is called fuzzy subhypermodule of M if the following conditions hold for all $x, y \in M$ and $r \in R$:

(i) $\mu(x) \wedge \mu(y) \leq \bigwedge_{\alpha \in x+y} \mu(\alpha)$;

(ii) $\mu(x) \leq \mu(-x)$;

(iii) $\mu(x) \leq \bigwedge_{z \in r \cdot x} \mu(z)$.

If μ is a fuzzy subhypermodule of M, then it is evident that, $\mu(-x) = \mu(x)$, and $\mu(x) \wedge \mu(y) \leq \bigwedge_{z \in x-y} \mu(z)$ for all $x, y \in M$.

Definition 2.7. A fuzzy subhypermodule μ of M is called normal if for all $x, y \in M$,

$$\mu(y) \leq \bigwedge_{\alpha \in x+y-x} \mu(\alpha).$$

Let M be an R-hypermodule. For a fuzzy subset μ of M, the level subset μ_t is defined as $\mu_t = \{x \in M : \mu(x) \geq t\}$, where $t \in [0, 1]$. A fuzzy subhypermodule can be characterized by its level subsets.

Proposition 2.8. *Let μ be a fuzzy subset of an R-hypermodule M. The following statements are equivalent:*

(i) *μ is fuzzy subhypermodule of M;*

(ii) *Every non-empty level subset of μ is a subhypermodule of M.*

Definition 2.9. Let μ be a normal fuzzy subhypermodule of M. We define the following equivalence relation on M:

$x \equiv y (mod\, \mu)$ if and only if there exists $\alpha \in x - y$ such that $\mu(\alpha) = \mu(0)$.

We denote the above equivalence relation by $\mu^\star$. Initially, one might think of $\mu^\star$ as defined by:

$$x \equiv y (mod\, \mu) \text{ if and only if } \bigvee_{\alpha \in x-y} \mu(\alpha) = \mu(0),$$

but in this case, $\mu^\star$ is only reflexive.

Let M be an R-hypermodule, and let ρ be an equivalence relation on M. We define the relation $\bar{\rho}$ on $P^*(M)$ as follows:

for $A, B \in P^*(M)$, we say $A\bar{\rho}B$ if and only if for every $a \in A$, there exists $b \in B$ such that $a\rho b$, and for every $b \in B$, there exists $a \in A$ such that $a\rho b$. An equivalence relation ρ on M is called regular if for every $x, y \in M$, $x\rho y$ implies $x + z\bar{\rho}y + z$ for all $z \in M$.

Lemma 2.10. [14] *The relation $\mu^\star$ is a regular relation.*

Let $x \in M$ and $\mu^\star[x]$ be the equivalence class of x and $\dfrac{M}{\mu} = \{\mu^\star[x] : x \in M\}$ be the set of all equivalence clasess of elements of M. We define the hyperoperations $\oplus$ and $\odot$ on $\dfrac{M}{\mu}$ as follows:

$$\mu^\star[x] \oplus \mu^\star[y] = \{\mu^\star[z] : z \in \mu^\star[x] + \mu^\star[y]\},$$
$$r \odot \mu^\star[x] = \{\mu^\star[z] : z \in r \cdot x\}.$$

Theorem 2.11. [14] *If M is an R-hypermodule, then $(\dfrac{M}{\mu}, \oplus, \odot)$ is an R-hypermodule.*

Lemma 2.12. *Let A be a normal subhypermodule of M and μ a normal fuzzy subhypermodule of M. Then $\mu_{|A}$ is a normal fuzzy subhypermodule of A.*

We note that the normalization of A in the above lemma is necessary to prove the normalization of $\mu_{|A}$.

Graph Theory Definitions

Next, we recall some important graph theory definitions required for our discussion:

Let X be a graph with the vertex set $V(X)$. The degree of a vertex v in X, denoted by $\deg(v)$, is the number of vertices adjacent to v. A graph is said to be connected if for each pair of distinct vertices v and w, there is a finite sequence of vertices $v_1, \cdots v_n$ such that each pair $\{v_i, v_{i+1}\}$ is an edge and $v_1 = v$, $v_n = w$. Such a sequence is called a path. A cycle is a path where $v = w$. The diameter of a graph X is defined as:

$$diam(X) = \bigvee_{a,b \in X} d(a, b).$$

where $d(a, b)$ is the length of the shortest path between vertices a and b.

A connected component in a graph is a set of vertices that are mutually connected. A complete graph is one in which every pair of distinct vertices is connected by an edge. A connected graph that contains no cycles is called a tree. A tree with n vertices is called a star if it has a vertex of degree $n-1$ and all other vertices have degree 1. A graph is Eulerian if there exists a closed path that uses every edge exactly once. A Hamiltonian graph is a graph that contains a cycle that visits each vertex exactly once.

3. Graphs Associated with R-hypermodules

In this section, we associate a graph with an R-hypermodule M using a normal fuzzy subhypermodule μ and a regular relation defined by $\mu^\star$ that is denoted by Γ_M^μ. The vertices of the graph correspond to the elements of M, and two vertices x and y are adjacent if $x\mu^\star y$ and is denoted by xEy. We explore several important properties of this graph, including connectedness, completeness, and Eulerian and Hamiltonian characteristics.

Example 3.1. Let $\mathbb{Z}$ be the ring of integers, and let $G = \{-1, 1\}$ be the normal subgroup of the multiplicative semigroup of $\mathbb{Z}$. According to [17], the quotient set $\bar{R} = \mathbb{Z}/G = \{0, G, 2G, 3G, \cdots\}$ forms a Krasner hyperring, with the hypersum and hyperproduct defined as:

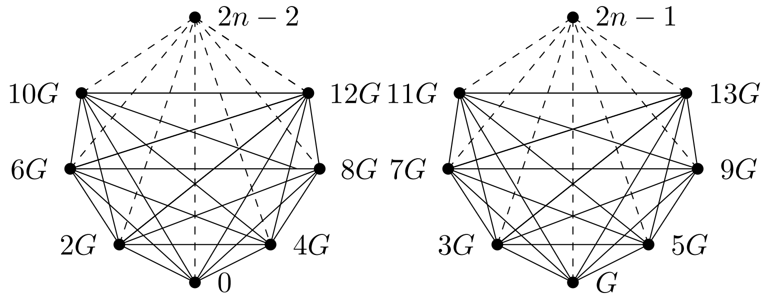
$$xG + yG = \{(xp + yq)G : p, q \in G\}, \;\; xG \cdot yG = xyG.$$

If we define hyperscalar product $xG \bullet yG = xG \cdot yG$, it is straightforward to verify that $(\bar{R}, +, \bullet)$ forms a $\bar{R}$-hypermodule over the Krasner hyperring $(\bar{R}, +, \cdot)$.

Define a normal fuzzy subhypermodule μ of $\bar{R}$ as:

$$\mu(x) = \begin{cases} 1 & x \in \{0, 2G, 4G, \cdots\} \\ 0 & x \in \{G, 3G, 5G, \cdots\} \end{cases}$$

For all $n \in \mathbb{N}$, it follows that $\mu^\star[(2n-2)G] = \mu^\star[2nG]$ and $\mu^\star[(2n-1)G] = \mu^\star[(2n+1)G]$. The graph associated with $\bar{R}$ is constructed as follows, with even-indexed elements grouped separately from odd-indexed ones:



Theorem 3.2. *Let μ and ν be normal fuzzy subhypermodules of M, with $\mu \subseteq \nu$ and $\mu(0) = \nu(0)$. Then Γ_M^μ is a subgraph of Γ_M^ν.*

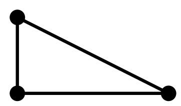
Proof. Let $\{x, y\} \in E(\Gamma_M^\mu)$. Then there exists $\alpha \in x - y$ such that $\mu(\alpha) = \mu(0)$. Since $\mu(\alpha) \leq \nu(\alpha)$, we have $\mu(0) \leq \nu(\alpha)$, completing the proof. $\square$

The converse of Theorem 3.2 is not true, as demonstrated by the following example:

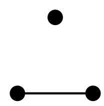
Example 3.3. Consider the ring $R = (\mathbb{Z}_4, +_4, \cdot_4)$ with the normal subgroup $G = \{1, 3\}$ from the multiplicative semigroup of $\mathbb{Z}_4$. As discussed in Example 3.1, $(\bar{R}, +, \bullet)$ forms a hypermodule over the Krasner hyperring $(\bar{R}, +, \cdot)$. Define two normal fuzzy subhypermodules μ and ν of $\bar{R}$ as follows:

$$\nu(0) = \nu(G) = \nu(2G) = 0, \quad \mu(x) = \begin{cases} 1 & x \in \{0, 2G\} \\ 0 & x = G \end{cases}$$

Clearly, $\Gamma_{\bar{R}}^\mu$ is subgraph of $\Gamma_{\bar{R}}^\nu$, but $\mu \not\subseteq \nu$ and $\nu(0) \neq \mu(0)$ The graphs $\Gamma_{\bar{R}}^\nu$ and $\Gamma_{\bar{R}}^\mu$ are depicted as follows:



$\Gamma_{\bar{R}}^\nu$



$\Gamma_{\bar{R}}^\mu$

As shown in [14] if, $f : M \to M'$ is a homomorphism of R-hypermodules and μ is a (normal) fuzzy subhypermodule of M, then $f(\mu)$ is a (normal) fuzzy subhypermodule of M', also if f is an epimorphism and ν is a (normal) fuzzy subhypermodule of M', then $f^{-1}(\nu)$ is a (normal) fuzzy subhypermodule of M.

Now, we have the following proposistion:

Proposition 3.4. *Let $f : M \to M'$ be an epimorphism of R-hypermodules, and μ and ν be normal fuzzy subhypermodules of M and M', respectively. Then:*

(i) *There exists a graph homomorphism from Γ_M^μ onto $\Gamma_{M'}^{f(\mu)}$.*
(ii) *There exists a graph homomorphism from $\Gamma_M^{f^{-1}(\nu)}$ onto $\Gamma_{M'}^\nu$.*

Proof. (i) Define a surjective map $\theta : \Gamma_M^\mu \to \Gamma_{M'}^{f(\mu)}$ by $\theta(m) = f(m)$. If $\{x, y\}$ is an edge of Γ_M^μ, then there exists $\alpha \in x - y$ such that $\mu(\alpha) = \mu(0)$. Since $f(\alpha) \in f(x) - f(y)$ and $f(\mu)(f(\alpha)) = \mu(\alpha) = \mu(0) = f(\mu)(0)$, it follows that $\{f(x), f(y)\} \in E(\Gamma_{M'}^{f(\mu)})$.

(ii) Similarly, define $\gamma : \Gamma_M^{f^{-1}(\nu)} \to \Gamma_{M'}^{f(\mu)}$ by $\gamma(m) = f(m)$. If $\{x, y\}$ is an edge of $\Gamma_M^{f^{-1}(\nu)}$, then there exists $\alpha \in x - y$ such that $f^{-1}(\nu)(\alpha) = f^{-1}(\nu)(0)$. Therefore $\nu(f(\alpha)) = \nu(0)$, and $f(x)\nu^\star f(y)$. $\square$

We note that for an R-hypermodule M, if μ is a normal fuzzy subhypermodule of M, the kernel of f is defined by:

$$Ker\, f = \{x \in M : f(x) = \mu^\star[0]\} = \{x \in M : \mu^\star[x] = \mu^\star[0]\},$$

where f is the natural epimorphism from M to $\dfrac{M}{\mu}$ and $\mu^\star[0]$ is the identity of the hypermodule $\dfrac{M}{\mu}$. It is also referred to as the heart of f and denoted by ω.

Theorem 3.5. *Let μ be a normal fuzzy subhypermodule of M. The graph Γ_M^μ is connected and complete if and only if the quotient $\dfrac{M}{\mu}$ is a trivial hypermodule and $\omega = M$*

Proof. Assume Γ_M^μ is connected. Then for every pair of vertices x, y, there exists a path from x to y, implying a sequence $x_0, \cdots, x_k$ of M such that, $x = x_0\mu^\star \cdots \mu^\star x_k = y$. Thus $x\mu^\star y$, leading to $\mu^\star = M \times M$. Consequently $\dfrac{M}{\mu}$ is a trivial hypermodule and $\omega = M$. Conversely, $|\dfrac{M}{\mu}| = 1$, then $\mu^\star = M \times M$, implying $x\mu^\star y$ for all $x, y \in M$. Hence Γ_M^μ is connected and complete. □

Corollary 3.6. *Let μ be a normal fuzzy subhypermodule of M. If Γ_M^μ is complete, then $diam(\Gamma_M^\mu) = 1$*

Proof. In a complete graph, every pair of vertices x and y is directly connected by an edge. Hence, the length of the shortest path between any two vertices is at most 1, implying $diam(\Gamma_M^\mu) = 1$. □

Theorem 3.7. *The connected components of Γ_M^μ are precisely the equivalence classes of the relation $\mu^\star$.*

Proof. Let K be a connected component of Γ_M^μ. For every pair of vertices x, y in K, there exists a path from x to y, implying a sequence $x_0, \cdots, x_k$ of M such that, $x = x_0\mu^\star \cdots \mu^\star x_k = y$. Thus $x\mu^\star y$, and it follows that $\mu^\star[x] = \mu^\star[y] = K$. □

Corollary 3.8. *Let μ be a normal fuzzy subhypermodule of M. Then, ω is a connected component of Γ_M^μ.*

Proof. This directly follows from Theorem 3.7. □

Proposition 3.9. *Let μ be a normal fuzzy subhypermodule of M. The degree of a vertex x in Γ_M^μ is equal to $|\mu^\star[x]|$, i.e, $d(x) = |\mu^\star[x]|$.*

Proof. The equivalence class $\mu^\star[x]$ represents the vertices adjacent to x. Thus, the degree of vertex x is the cardinality of $\mu^\star[x]$. □

Definition 3.10. A graph is called regular if all its vertices have the same degree. If each vertex has degree k, the graph is referred to as k-regular.

By Proposition 3.9, if $\mu^\star[x] = k$ for all $x \in M$, then Γ_M^μ is a k-regular graph.

Recall that a graph X is called bipartite if its vertex set can be devided into two disjoint sets V_1 and V_2 such that every edge of X connects a vertex in V_1 with a vertex in V_2. We call V_1, V_2 a bipartition of $V(X)$.

Lemma 3.11. *Let μ be a normal fuzzy subhypermodule of M, and M_1 and M_2 are non-empty subsets of M. Denote by $\Gamma_M^\mu = (M_1, M_2, E)$, the bipartite graph associated with M. If $x_i, y_i \in M_i$, $i = 1, 2$, then for all $\alpha \in x_i - y_i$, we have $\mu(\alpha) \neq \mu(0)$.*

Proof. Assume there exists $\alpha \in x_i - y_i$ for some $x_i, y_i \in M_i$, such that, $\mu(\alpha) = \mu(0)$. This would imply $x_i \mu^\star y_i$, contradicting the assumption that x_i and y_i belong to different parts of the bipartition. Therefore, $\mu(\alpha) \neq \mu(0)$. □

Lemma 3.12. [14] *Let $f : M \longrightarrow M'$ be a homomorphism of R-hypermodules, and let ν be fuzzy subset of M'. If f is an epimorphism, then $f(f^{-1}(\nu)) = \nu$.*

Theorem 3.13. *Let μ and ν be normal fuzzy subhypermodules of M and M', respectively with $|\omega| = 1$ and $M \cong M'$. Then, Γ_M^μ and $\Gamma_{M'}^\nu$ are isomorphic graphs.*

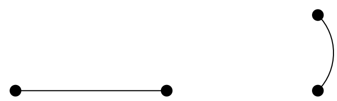
Proof. Since $M \cong M'$, there exists an isomorphism $f : M \longrightarrow M'$. If $x\mu^\star y$, then there exists $\alpha \in x - y$ such that $\mu(\alpha) = \mu(0)$, which implies $\alpha\mu^\star 0$. Given that $|\omega| = 1$, it follows that $\alpha = 0$. Since f is an isomorphism and $f(0) = 0$, we have $f(\alpha) = 0 \in f(x - y)$. Therefore $f(x)\nu^\star f(y)$, which shows that $\{f(x), f(y)\} \in E(\Gamma_{M'}^\nu)$. Thus, the graph homomorphism induced by f preserves adjacency, proving that Γ_M^μ and $\Gamma_{M'}^\nu$ are isomorphic. □

The converse of Theorem 3.13 is not true, as shown by the following example:

Example 3.14. Consider the rings $R = (\mathbb{Z}_3, +_3, \cdot_3)$ and $S = (\mathbb{Z}_2, +_2, \cdot_2)$, with normal subgroups $G = \{1, 2\}$ and $G' = \{1\}$ from the multiplicative semigroup of $\mathbb{Z}_3$ and $\mathbb{Z}_2$, respectively. As demonstrated in Example 3.1, the quotient sets $\bar{R}$ and $\bar{S}$ form hypermodules over the Krasner hyperrings $\mathbb{Z}_3$ and $\mathbb{Z}_2$, respectively.
Define the following normal fuzzy subhypermodules μ and ν of $\bar{R}$ and $\bar{S}$, respectively by:

$$\mu(0) = \mu(G) = 1, \;\; \nu(0) = \nu(G) = 0.$$

Clearly, the graphs $\Gamma_{\bar{R}}^\mu$ and $\Gamma_{\bar{S}}^\nu$ are isomorphic, and their representations are:



It is straightforward to verify that $\bar{R} \ncong \bar{S}$, as their algebraic structures differ despite the isomorphism of their associated graphs, $\Gamma_\mu^{\bar{R}} \cong \Gamma_\nu^{\bar{S}}$.

This example demonstrates that although the graphs $\Gamma_{\bar{R}}^\mu$ and $\Gamma_{\bar{S}}^\nu$ are isomorphic, the hypermodules $\bar{R}$ and $\bar{S}$ are not isomorphic.

Remark 3.15. The condition $|\omega| = 1$ of Theorem 3.13, is necessary. For example, consider the R-hypermodule in Example 3.3, where $\omega = \{0, 2G\}$. In this case, $\{0, G\} \in E(\Gamma_{\bar{R}}^\nu)$, but $\{0, G\} \notin E(\Gamma_{\bar{R}}^\mu)$. So $\Gamma_{\bar{R}}^\mu \ncong \Gamma_{\bar{R}}^\nu$.

Theorem 3.16. *Let A be a normal subhypermodule of M and μ a normal fuzzy subhypermodule of M. Then $\Gamma_A^{\mu_{|A}}$ is a subgraph of Γ_M^μ.*

Proof. By Lemma 2.12, $\mu_{|A}$ is a normal fuzzy subhypermodule of A. Since A is a subhypermodule of M, the vertices of $\Gamma_A^{\mu_{|A}}$ are a subset the vertices of Γ_M^μ. If $x\mu_{|A}^\star y$ then, $x\mu_M^\star y$ and therefore $\Gamma_A^{\mu_{|A}}$ is a subgraph of Γ_M^μ. □

Lemma 3.17. *Let μ be a normal fuzzy subhypermodule of M. Then each non-empty level subset of μ is a normal subhypermodule of M.*

Corollary 3.18. *Let μ be a normal fuzzy subhypermodule of M. Then $\Gamma_{\mu_t}^{\mu|_{\mu_t}}$ is a subgraph of Γ_M^{μ}.*

Remark 3.19. Let μ be a normal fuzzy subhypermodule of M. There is no direct correspondence between the subgraphs of Γ_M^{μ} and the normal fuzzy subhypermodules of M that are contained within μ. For example, consider the ring $(\mathbb{Z}_3, +_3, \cdot_3)$. As is shown in Example 3.14, $(\bar{R}, +, \bullet)$ forms a hypermodule over the Krasner hyperring $(\bar{R}, +, \cdot)$, and the number of subgraphs $\Gamma_{\bar{R}}^{\mu}$ is 4, while the number of normal fuzzy subhypermodules of $\bar{R}$ is 1. This hilights that there is no one-to-one correspondence between subgraphs of Γ_M^{μ} and normal fuzzy subhypermodules of M contained whitin μ.

Let μ be a normal fuzzy subhypermodule of M. There is no direct one-to-one correspondence between subgraphs of Γ_M^{μ} and the non-empty level subsets of μ. For instance, in Example 3.14, the number of subgraphs of $\Gamma_{\bar{R}}^{\mu}$ is 4, whereas the number of level subsets of μ is 1.

Theorem 3.20. *Let μ be a normal fuzzy subhypermodule of M. Then Γ_M^{μ} is Eulerian if and only if for all $x \in M$, $|\mu^{\star}[x]|$ is an odd number and Γ_M^{μ} is a connected graph.*

Proof. First, assume that Γ_M^{μ} is Eulerian. This means that Γ_M^{μ} is connected and complete, and all vertices must have even degree. However, in the associated graph of an R-hypermodule, each vertex has a loop, implying the degree of each vertex must be odd. Therefore, $|\mu^{\star}[x]|$ must be odd for every $x \in M$, and the graph is connected. Conversely, if Γ_M^{μ} is connected and $|\mu^{\star}[x]|$ is odd for all $x \in M$, then Γ_M^{μ} is a complete graph where every vertex has an odd degree. Complete graphs with all vertices of odd degree are Eulerian, hence Γ_M^{μ} is Eulerian. □

Lemma 3.21. *For an R-hypermodule M, let μ be a normal fuzzy subhypermodule of M such that $|M| \geq 3$ and $|\dfrac{M}{\mu}| = 1$. Then Γ_M^{μ} is a Hamiltonian graph. Moreover, if $|M|$ is odd, Γ_M^{μ} is both Eulerian and Hameltonian.*

Proof. By Theorem 3.5, Γ_M^{μ} is a complete graph. It is well-known that complete graphs with more than three vertices are Hamiltonian. Additionally, the presence of loops in Γ_M^{μ} does not affect the existence of a Hamiltonian path, as the loops can be bypassed. Therefore, Γ_M^{μ} is Hamiltonian. Furthermore, if $|M|$ is odd, every vertex in the complete graph (including loops) will have an odd degree, satisfying the condition for Eulerian graphs. Hence, Γ_M^{μ} is both Eulerian and Hamiltonian. □

It is known that in each vertex of the graph associated with an R-hypermodule has a loop. If we consider this graph without loops, we arrive at the following lemma.

Lemma 3.22. *For an R-hypermodule M, let μ be a normal fuzzy subhypermodule of M with $|M| \geqslant 3$, and assume Γ_M^{μ} is connected. Then, Γ_M^{μ} is not a tree.*

Proof. Since Γ_M^{μ} is connected and complete, it necessarily contains cycles. A tree, by definition, is a connected graph with no cycles. Therefore, Γ_M^{μ} cannot be a tree. □

Remark 3.23. For an R-hypermodule M, if $|M| \geq 3$, then Γ_M^μ is not a star graph. If Γ_M^μ were a star graph and e were a vertex connected to other vertices of Γ_M^μ, then $\mu^\star$ would be an equivalence relation, implying $x\mu^\star y$ for all $x, y \in M$, which contradicts the assumption that Γ_M^μ is a star graph, for instance, in Example 3.14, $\bar{R}$ is shown to be a hypermodule, and $|M| < 3$ prevents the graph from being a star graph as follows:

Lemma 3.24. *Let* $|\omega| = 1$. *Then* Γ_M^μ *is niether Hamiltonian nor Eulerian.*

Proof. Since $|\omega| = 1$, the graph Γ_M^μ is not connected. A Hamiltonian graph must contain a cycle that visits every vertex exactly once, and an Eulerian graph must have a closed path that uses each edge exactly once, both of which require the graph to be connected. Therefore, Γ_M^μ is neither Hamiltonian nor Eulerian. □

4. Conclusion

In this study, we introduced a framework for associating graphs with R-hypermodules through normal fuzzy subhypermodules. By leveraging the concept of regular relations, we examined how algebraic properties of these structures manifest in their corresponding graphs. Key characteristics such as connectivity, completeness, Hamiltonian and Eulerian properties were rigorously analyzed, providing a deeper understanding of the interplay between fuzzy algebraic structures and graph theory. Our findings underscore the significance of normal fuzzy subhypermodules in shaping the structural attributes of the associated graphs. Notably, we demonstrated that conditions such as the triviality of certain submodules directly impact the graphs ability to exhibit Hamiltonian or Eulerian behavior. This connection enriches the study of hypermodules by offering a new perspective that bridges algebraic hyperstructures and combinatorial graph properties. Beyond theoretical implications, these insights pave the way for practical applications in areas such as network design, optimization, and fuzzy systems. The ability to model complex relationships using fuzzy algebraic structures and their graphical counterparts holds potential for advancements in fields ranging from computer science to data analysis and engineering. Future research could expand on this work by exploring other types of fuzzy hyperstructures and investigating their graphical representations in more complex networks. Additionally, studying dynamic changes in such graphs could lead to new discoveries in evolving systems and adaptive networks. In conclusion, this study not only advances the theoretical landscape of fuzzy algebraic structures but also opens doors to novel applications in interdisciplinary contexts, emphasizing the versatility and power of combining algebra with graph theory.

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