DOI: 10.22098/jhs.2025.16534.1070

Journal of Hyperstructures, 14 (1) (2025), 16-24

Journal Homepage: https://jhs.uma.ac.ir/

Research Paper

CONGRUENCE RELATION IN FUZZY PARTIAL HYPERALGEBRAS

ELHAM SADAT HOSEINI[1], TAHERE NOZARI[2,*], REZA AMERI [3] AND BIBI NAIMEH ONAGH [4]

[1] Department of Mathematics, Faculty of Sciences, Golestan University, Gorgan, Iran, e.hoseini99@stu.gu.ac.ir

[2] Department of Mathematics, Faculty of Sciences, Golestan University, Gorgan, Iran, t.nozari@gu.ac.ir

[3] School of Mathematics, Statistics and Computer Science, College of Sciences, University of Tehran, Tehran, Iran, rameri@ut.ac.ir

[4] Department of Mathematics, Faculty of Sciences, Golestan University, Gorgan, Iran, bn.onagh@gu.ac.ir

ARTICLE INFO

Article history:
Received: 12 January 2025
Accepted: 21 May 2025
Communicated by Yuming Feng

Keywords:
Compatible relation
Congruence relation
Fuzzy hyperalgebra
Fuzzy partial hyperalgebra

MSC:
08A72; 03E72; 08A55; 08A30

ABSTRACT

In this paper, we begin by introducing the concept of fuzzy partial hyperalgebra and exploring the relationships between congruence relations and strong congruence relations within this framework. We then construct an embedding of any fuzzy partial hyperalgebra into a fuzzy hyperalgebra, ensuring that all congruence relations on the embedded fuzzy partial hyperalgebra can be simultaneously extended to the corresponding fuzzy hyperalgebra.

1. Introduction

The concept of hyperstructures was first introduced by Marty [12] in 1934 during the 8th Congress of Scandinavian Mathematicians. In 1965, Zadeh [17] introduced the idea of fuzzy sets, which has since led to the development of numerous fuzzy algebraic structures. Consequently, researchers have devoted significant attention to exploring and studying the

*Address correspondence to T. Nozari; Department of Mathematics, Golestan University, Gorgan, Iran, E-mail: t.nozari@gu.ac.ir.

This work is licensed under a Creative Commons Attribution-NonCommercial 4.0 International License.
Copyright © 2025 The Author(s). Published by University of Mohaghegh Ardabili.

fundamental concepts of fuzzy hyperstructures. Corsini and Leoreanu investigated the ordinary operations on fuzzy sets in their work [5]. For a non-empty set H, an ordinary operation maps each pair of elements in H to a non-empty subset of H, while a fuzzy hyperoperation associates a fuzzy subset with each pair of elements in H. This concept was first introduced in [6]. In [3], Ameri and Zahedi presented the concept of hyperalgebraic systems Schweigert investigated the congruence relations of multialgebras in [14]. In [2], Ameri and Rosenberg, building on the work in [3], introduced the concepts of compatibility, strong compatibility, congruences, and strong congruences within hyperalgebras, thereby extending the notion of congruences found in ordinary algebras. They also introduced the lattices of congruences and strong congruences in hyperalgebras, illustrating that the characteristics of these lattices are distinct from those found in ordinary algebras. Sen, Ameri, and Chowdhury [15] introduced an innovative approach to fuzzy semihypergroups. This approach was subsequently extended to include the concepts of fuzzy hyperrings [9] and fuzzy hypermodules [10]. Building on [15], Ameri and Nozari expanded this approach to encompass hyperalgebras, the most extensive category of fuzzy hyperalgebraic systems in [4]. They introduced and analyzed the concept of fuzzy hyperalgebras and their relationships with hyperalgebras, establishing a connection between fuzzy hyperalgebras and ordinary algebras.

This paper introduces fuzzy partial hyperalgebras to study subsets H of a hyperalgebra $\mathbb{K}$, even if H is not closed under fuzzy hyperoperations. By removing n_i-tuples from each n_i-ary hyperoperation β_i that produce values outside H, we build a simpler structure for analysis. Inspired by partial algebras [8], this approach simplifies the study of fuzzy hyperstructures [7]. We focus on congruence relations and their links to strong congruence, building on recent studies [1, 13]. This work offers a clear framework for these relationships, supporting further research into algebraic properties like distributivity [16].

2. Preliminaries

In this section, the key definitions and preliminary findings are introduced that will be referenced in the following sections.

Recall that for a non-empty set H, a fuzzy subset μ of H is a function from H into the real unit interval $[0, 1]$. μ is a non-zero fuzzy subset of H if there exists some $x \in H$ such that $\mu(x) > 0$.

For two fuzzy subsets μ and ν of H, we say μ is smaller than ν, and write $\mu \leq \nu$ if for all $x \in H$, $\mu(x) \leq \nu(x)$.

A fuzzy n-ary hyperoperation f^n on a non-empty set H is a map $f^n : H \times H \times \cdots \times H \to \mathcal{F}^*(H)$ which associates a non-zero fuzzy subset $f^n(a_1, \ldots, a_n)$ with any n-tuple $(a_1, \ldots, a_n)$ of elements of H, where $\mathcal{F}^*(H)$ is the set of all non-zero fuzzy subsets of H ([4]).

Let H be a non-empty set and for every $i \in I$, let β_i be a fuzzy n_i-ary hyperopertion on H. Then $\mathbb{H} = \langle H, (\beta_i |\ i \in I)\rangle$ is called a fuzzy hyperalgebra, where $(n_i : i \in I)$ is the type of this fuzzy hyperalgebra. Two fuzzy hyperalgebras are similar if they have the same type ([4]).

A non-empty subset S of a fuzzy hyperalgebra $\mathbb{H}$ is a fuzzy subhyperalgebra of $\mathbb{H}$ if for all $i \in I$, and for all $a_1, \ldots, a_{n_i} \in S$, the following condition holds [4]:

$$\text{if } \beta_i(a_1, \ldots, a_{n_i})(x) > 0 \text{ then } x \in S.$$

A binary relation on a fuzzy hyperalgebra $\mathbb{H}$ is a subset of H^2. A binary relation ρ on H is said to be an equivalence relation if [2]:
(i) $(x,x) \in \rho$ for all $x \in H$ (reflexive),
(ii) if $(x,y) \in \rho$ then $(y,x) \in \rho$ for all $x,y \in H$ (symmetric),
(iii) if $(x,y) \in \rho$ and $(y,x) \in \rho$, then $(x,z) \in \rho$ for all $x,y,z \in H$ (transitive).

Definition 2.1. [4] Let ρ be an equivalence relation on a hyperalgebra $\mathbb{H} = \langle H, (\beta_i |\ i \in I)\rangle$ and μ and ν be two fuzzy subsets on H. We say that $\mu\overline{\rho}\nu$ if following two conditions hold:
(i) For all $a \in H$, if $\mu(a) > 0$, there exists $b \in H$ such that $\nu(b) > 0$ and $(a,b) \in \rho$,
(ii) For all $b \in H$, if $\nu(b) > 0$, there exists $a \in H$ such that $\mu(a) > 0$ and $(a,b) \in \rho$,
and we say $\mu\overline{\overline{\rho}}\nu$, if for every $a,b \in H$ such that $\mu(a) > 0$ and $\nu(b) > 0$, then $(a,b) \in \rho$.

3. Congruence Relations In Fuzzy Partial Hyperalgebras

In this section, we introduce the concepts of fuzzy partial hyperalgebra and relative fuzzy subhyperalgebra of the fuzzy partial hyperalgebra. We also extend the congruence relation on a fuzzy partial hyperalgebra to a congruence relation on another fuzzy hyperalgebra that contains the first set as a relative fuzzy subhyperalgebra.

Definition 3.1. A binary relation ρ on a fuzzy hyperalgebra $\mathbb{H} = \langle H, (\beta_i |\ i \in I)\rangle$ is called compatible with β_i if for every $i \in I$ and for all $a_1, \ldots, a_{n_i}, b_1, \ldots b_{n_i} \in H$ such that $a_1\rho b_1, \ldots, a_{n_i}\rho b_{n_i}$, we have

$$\beta_i(a_1, \ldots, a_{n_i})\ \overline{\rho}\ \beta_i(b_1, \ldots, b_{n_i}).$$

An equivalence relation ρ on fuzzy hyperalgebra $\mathbb{H}$ is called congruence on $\mathbb{H}$ if it is compatible with every β_i, for all $i \in I$.

A binary relation ρ on a fuzzy hyperalgebra $\mathbb{H} = \langle H, (\beta_i \mid i \in I)\rangle$ is called a strongly compatible with β_i if for every $i \in I$ and for all $a_1, \ldots, a_{n_i}, b_1, \ldots b_{n_i} \in H$ such that $a_1\rho b_1, \ldots, a_{n_i}\rho b_{n_i}$, we have

$$\beta_i(a_1, \ldots, a_{n_i})\ \overline{\overline{\rho}}\ \beta_i(b_1, \ldots, b_{n_i}).$$

An equivalence relation ρ on $\mathbb{H}$ is called strongly congruence relation on $\mathbb{H}$ if it is a strongly compatible with every β_i, for all $i \in I$.

Denote by $Con(\mathbb{H})$ ($Cons(\mathbb{H})$) the set of all congruences (strong congruences) of $\mathbb{H}$.

Clearly, every strongly congruence of a fuzzy hyperalgebra is a congruence too.

Next, to illustrate the concept of strong congruence more clearly, we present an example of a strong congruence relation on a fuzzy hyperalgebra and verify its compatibility with the corresponding operations.

Example 3.2. Let $\mathbb{H} = \langle H = \{a,b,c\}, \beta\rangle$ be a fuzzy hyperalgebra with binary hyperoperations defined as shown in the following table. If $\rho = \{(a,a),(b,b),(c,c),(a,b),(b,a)\}$, then ρ is a strong congruence relation on $\mathbb{H}$.

H \ β	$\beta(a,a)$	$\beta(a,b)$	$\beta(a,c)$	$\beta(b,a)$	$\beta(b,b)$	$\beta(b,c)$	$\beta(c,a)$	$\beta(c,b)$	$\beta(c,c)$
a	0.8	0.6	0	0.6	0.3	0	0	0	0
b	0.2	0.4	0	0.4	0.7	0	0	0	0.3
c	0	0	0	0	0	0	0	0	0.7

The relation ρ satisfies the conditions of strong congruence with respect to the operation β, as verified in all cases. This confirms the compatibility of the operation with the given equivalence relation.

Definition 3.3. Let $\mathbb{H} = \langle H, (\beta_i |\ i \in I)\rangle$ be a fuzzy hyperalgebra, $\rho \in Con(\mathbb{H})$ and $a \in H$. The congruence class determined by a and ρ, denoted by ρ_a, is the subset of H defined by:

$$\rho_a = \{b \in H |\ (a,b) \in \rho\}.$$

Definition 3.4. A fuzzy partial hyperalgebra is a pair $\mathbb{H} = \langle H, (\beta_i |\ i \in I)\rangle$, where for each $i \in I$, there are an ordinal number n_i and a set $X_i \subseteq H^{n_i}$ such that β_i is a function from X_i into $\mathcal{F}^*(H)$. In case $X_i = H^{n_i}$ the fuzzy hyperoperation β_i is called total fuzzy hyperoperation, and otherwise, it is called a fuzzy partial hyperoperation on $\mathbb{H}$.

Definition 3.5. Let $\mathbb{H} = \langle H, (\beta_i |\ i \in I)\rangle$ be a fuzzy partial hyperalgebra. If β_i, where $i \in I$, is a n_i-ary fuzzy partial hyperoperation, then $D(\beta_i, \mathbb{H})$ denotes the domain of β_i in $\mathbb{H}$, that is, $D(\beta_i, \mathbb{H})$ is a subset of H^{n_i}.

Definition 3.6. Let $\mathbb{H}' = \langle H', (\beta_i |\ i \in I)\rangle$ be a fuzzy partial hyperalgebra. Then $\mathbb{H}'$ is a fuzzy subhyperalgebra of the fuzzy partial hyperalgebra $\mathbb{H} = \langle H, (\beta_i |\ i \in I)\rangle$ if $H \subseteq H'$. In this case, β_i on $\mathbb{H}'$ is the restriction of β_i on $\mathbb{H}$ to H'.

Remark 3.7. According to the above definition, we have

$$D(\beta_i, \mathbb{H}') = D(\beta_i, \mathbb{H}) \cap H'^{n_i}$$

Definition 3.8. The fuzzy partial hyperalgebra $\mathbb{H}' = \langle H', (\beta_i |\ i \in I)\rangle$ is a relative fuzzy subhyperalgebra of the fuzzy partial hyperalgebra $\mathbb{H} = \langle H, (\beta_i |\ i \in I)\rangle$, if $H' \subseteq H$ and for all β_i, $i \in I$,

$$D(\beta_i, \mathbb{H}') = \{(a_1, \ldots, a_{n_i}) |\ (a_1, \ldots, a_{n_i}) \in D(\beta_i, \mathbb{H}) \cap H'^{n_i}, \beta_i(a_1, \ldots, a_{n_i}) \in \mathcal{F}^*(H')\}.$$

We also say that $\mathbb{H}'$ is embedded in $\mathbb{H}$.

Definition 3.9. Let $\mathbb{H} = \langle H, (\beta_i |\ i \in I)\rangle$ and $\mathbb{H}' = \langle H', (\beta_i |\ i \in I)\rangle$ be two fuzzy partial hyperalgebras such that $\mathbb{H}'$ is embedded in $\mathbb{H}$ and θ is a congruence relation on $\mathbb{H}$. Denote the restriction of θ to $\mathbb{H}'$ by $\theta_{H'}$.

Definition 3.10. Let $\mathbb{H} = \langle H, (\beta_i |\ i \in I)\rangle$ and $\mathbb{K} = \langle K, (\beta_i |\ i \in I)\rangle$ be two fuzzy partial hyperalgebras and $H \subseteq K$. Let θ be a congruence relation on $\mathbb{H}$. Then θ can be extended to a relation ρ on $\mathbb{K}$, if ρ is a congruence relation on $\mathbb{K}$ and $\theta \subseteq \rho$, i.e., for every $a, b \in H$, $(a,b) \in \theta$ implies $(a,b) \in \rho$.

Definition 3.11. let $\mathbb{H} = \langle H, (\beta_i |\ i \in I)\rangle$ be a fuzzy partial hyperalgebra with the congruence relation θ. A fuzzy subhyperalgebra $\mathbb{H}'$ is called θ-strong if $\theta_{H'}$ is a strong congruence relation on $\mathbb{H}'$.

Clearly, $\mathbb{H}$ is a fuzzy θ-strong subhyperalgebra of $\mathbb{H}$ if and only if θ is a strong congruence relation on $\mathbb{H}$.

In the following theorem, we will state the main objective of the article. We show that a fuzzy partial hyperalgebra can be embedded into a fuzzy hyperalgebra in such a way that a specific congruence relation on the latter corresponds to a given congruence relation on the former.

Theorem 3.12. *Let $\mathbb{H} = \langle H, (\beta_i|\ i \in I)\rangle$ be a fuzzy partial hyperalgebra and θ be a congruence relation on $\mathbb{H}$. Then $\mathbb{H}$ can be embedded into a fuzzy hyperalgebra $\mathbb{K}$ so that*
(i) there is a congruence relation ρ on $\mathbb{K}$ such that $\rho_H = \theta$,
(ii) for every fuzzy θ-strong subhyperalgebra $\mathbb{H}'$ of $\mathbb{H}$, there is a fuzzy subhyperalgebra $\mathbb{K}'$ of $\mathbb{K}$ extending $\mathbb{H}'$ such that $(\theta_{H'})_a = \phi_H$, for all $a \in H'$ and some congruence relation ϕ on $\mathbb{K}'$.

Proof. (i) Let $\mathbb{H} = \langle H, (\beta_i|\ i \in I)\rangle$ be a fuzzy partial hyperalgebra and θ be a congruence relation on $\mathbb{H}$ and

$$B_1, B_2, \ldots, B_i, \ldots\ i < \alpha,$$

be the congruence classes of θ, with α being a finite number.
Put

$$A := H \cup \{b_i\} \cup \{m\},\ i < \alpha,$$

where $b_t \neq b_s$, if $t \neq s$ $(s, t < \alpha)$, and $b_i \neq m$, $b_i \notin H$ for all $i < \alpha$ and $m \notin H$.
Let

$$\{m\}, B_1 \cup \{b_1\}, B_2 \cup \{b_2\}, \ldots, B_i \cup \{b_i\}, \ldots,\ i < \alpha,$$

be the blocks of a partition $\overline{B}$ of A. Put

$$B'_i := B_i \cup \{b_i\},\ i < \alpha.$$

Define β_i^A for $(a_1, \ldots, a_{n_k}) \in A^{n_i}$ as follows:
1) if $(a_1, \ldots, a_{n_i}) \in B_1 \times \cdots \times B_{n_i} \cap D(\beta_i, \mathbb{H})$,

$$\beta_i^A(a_1, \ldots, a_{n_i}) = \beta_i^H(a_1, \ldots, a_{n_i}).$$

2) if

$$(a_i, \ldots, a_{n_i}) \in B'_1 \times \cdots \times B'_{n_i} \backslash D(\beta_i, \mathbb{H}),$$

and if there is some

$$(a'_1, \ldots, a'_{n_i}) \in B_1 \times \cdots \times B_{n_i} \cap D(\beta_i, \mathbb{H}),$$

and $s < \alpha$ such that $\beta_i^A(a_1, \ldots, a_{n_k}) \in \mathcal{F}^*(B_s)$, then for every $z \in A$,

$$\beta_i^A(a_1, \ldots, a_{n_i})(z) = \chi_{\cup\{b_s\}}(z).$$

3) if $a_j = m$ for some $j = 1, \ldots, n_i$ or if

$$(a_1, \ldots, a_{n_i}) \in B'_1 \times \cdots \times B'_{n_i}$$

and

$$B_1 \times \ldots \times B_{n_i} \cap D(\beta_i, \mathbb{H}) = \emptyset,$$

then for every $z \in A$

$$\beta_i^A(a_1, \ldots, a_{n_i})(z) = \chi_{\{m\}}(z).$$

This definition makes each β_i^A a fuzzy hyperoperation on A and, $\mathbb{K} = \langle A, (\beta_i^A|\ i \in I)\rangle$ a fuzzy hyperalgebra. Part (1) of the above definition shows that $\mathbb{H}$ is a relative fuzzy subhyperalgebra of $\mathbb{K}$, since (2) and (3) produce elements outside of $\mathbb{H}$. The partition $\overline{B}$ induces an equivalence relation ρ on $\mathbb{K}$. Also

$\{m\}$, B_1', ..., B_i', ..., $i < \alpha$, are the equivalence classes of ρ,

$\{m\} \cap H = \emptyset$, $B_1' \cap H = B_1$, ..., $B_i' \cap H = B_i$, ..., $i < \alpha$,

θ induces the congruence classes $B_1, \ldots, B_i, \ldots, i < \alpha$, on H. Therefore $\rho_H = \theta$.

Now we must to show ρ is a congruence relation on $\mathbb{K}$. For every $i = 1, \ldots, n_i$, let $a_i, b_i \in H$ such that $(a_i, b_i) \in \rho$. Note that
(a) if for some $j = 1, \ldots, n_i$, $a_j = m$, then $b_j = m$ and hence by part (3),

$$\beta_i^A(a_1, \ldots, a_{n_i}) = \beta_i^A(b_1, \ldots, b_{n_i}) = \chi_{\{m\}}.$$

So, if $\beta_i^A(a_1, \ldots, a_{n_i})(z) > 0$, $z \in A$, we have $\chi_{\{m\}}(z) > 0$ and $z = m$. Then there exists $t = m$ such that $\beta_i^A(b_1, \ldots, b_{n_i})(t) > 0$. Since $m = z = t$, $(t, z) \in \rho$ and ρ is congruence relation on $\mathbb{K}$.

(b) if $(a_1, \ldots, a_{n_i}) \in B_1' \times \cdots \times B_{n_i}'$, then for every $i = 1, \ldots, n_i$, $a_i \in B_i' = B_i \cup \{b_i\}$. Since $(a_i, b_i) \in \rho$, $b_i \in B_i' = B_i \cup \{b_i\}$ and

$$(b_1, \ldots, b_{n_i}) \in B_1' \times \cdots \times B_{n_i}'.$$

If $B_1 \times \cdots \times B_{n_i} \cap D(\beta_i, \mathbb{H}) = \emptyset$, then by part (3) of the definition,

$$\beta_i^A(a_1, \ldots, a_{n_i}) = \beta_i^A(b_1, \ldots, b_{n_i}) = \chi_{\{m\}},$$

and according to the part (a), ρ is congruence relation on $\mathbb{K}$.

If $B_1 \times \cdots \times B_{n_i} \cap D(\beta_i, \mathbb{H}) \neq \emptyset$, three situations occur:

(S_1) both $\beta_i^A(a_1, \ldots, a_{n_i})$ and $\beta_i^A(b_1, \ldots, b_{n_i})$ are $\chi_{\cup\{b_s\}}$, for some $s < \alpha$. If $\beta_i^A(a_1, \ldots, a_{n_i})(z) > 0$, $z \in A$, then $z \in \cup\{b_s\}$. So, there exists $1 \leq j' \leq n_i$ such that $z \in B_{j'} \cup \{b_{j'}\}$ and $z = b_{j'}$. In this case, if $t = b_{j'}$, then $(t, z) \in \rho$ and ρ is congruence relation on $\mathbb{K}$.

(S_2) one of them, say $\beta_i^A(a_1, \ldots, a_{n_i})$, is in $\mathcal{F}^*(B_s)$, for some $s < \alpha$, and the other is $\chi_{\cup\{b_s\}}$. For every $z \in A$ such that $\beta_i^A(a_1, \ldots, a_{n_i})(z) > 0$, there exists $r < \alpha$ such that $z \in B_r' = B_r \cup \{b_r\}$. Consider $t = b_r$; then $\beta_i^A(b_1, \ldots, b_{n_i})(t) > 0$ and $(t, z) \in \rho$. Thus ρ is a congruence relation on $\mathbb{K}$.

(S_3) both $\beta_i^A(a_1, \ldots, a_{n_i})$ and $\beta_i^A(b_1, \ldots, b_{n_i})$ are in $D(\beta_i, \mathbb{H})$, then by part (1) of the definition,

$$\beta_i(a_1, \ldots, a_{n_i}) \ \overline{\theta}\ \beta_i(b_1, \ldots, b_{n_i}).$$

Sine $\rho_H = \theta$,

$$\beta_i^A(a_1, \ldots, a_{n_i}) \ \overline{\rho}\ \beta_i^A(b_1, \ldots, b_{n_i}),$$

and ρ is congruence relation on $\mathbb{K}$. So, the proof of (i) is complete.

(ii) Let $\mathbb{H}' = \langle H', (\beta_i| \ i \in I)\rangle$ be a fuzzy θ-strong subhyperalgebra of $\mathbb{H}$. Then $\theta_{H'}$ is a strong congruence relation on $\mathbb{H}'$. Put $C_i := B_i \cap H'$ and $A' = H' \cup \{b_i\} \cup \{m\}$, where $i < \alpha$. Then $A' \cap B_i = C_i$. Now consider $\mathbb{K}' = \langle A', (\beta_i^A| \ i \in I)\rangle$. We must to show $\mathbb{K}'$ is fuzzy subhyperalgebra of $\mathbb{K}$. Let $\beta_i^A(a_1, \dots, a_{n_i})(z) > 0$, where $a_1, \dots, a_{n_i} \in A'$. Then

$(FS1)$ if $C_1 \times \cdots \times C_{n_i} \cap D(\beta_i, \mathbb{H}) = \emptyset$, by part (3) of the definition in (i), $z = m$ and $z \in A'$;

$(FS2)$ if $C_1 \times \cdots \times C_{n_i} \cap D(\beta_i, \mathbb{H}) \neq \emptyset$, since $A' \cap B_i = C_i$,

$$C_1 \times \cdots \times C_{n_i} \cap D(\beta_i, \mathbb{H}) = C_1 \times \cdots \times C_{n_i},$$

by part (2) of the definition in (i), $z \in \cup\{b_s\}$ for $s < \alpha$, so $z \in A'$;

$(FS3)$ if $(a_1, \dots, a_{n_i}) \in C_1 \times \cdots \times C_{n_i} \backslash D(\beta_i, \mathbb{H})$, by part (2) of the definition in (i), it will be similar to $(FS2)$.

Obviously $\mathbb{H}'$ is a fuzzy relative subhyperalgebra of $\mathbb{K}'$. Also $C_i = B_i \cap H'$ are the congruence classes of $\theta_{H'}$.

On the other hand,

$$\{b_1\}, \{b_2\}, \dots, \{b_i\}, \dots, \{m\}, C_1, \dots, C_i, \dots, \ i < \alpha,$$

define a congruence class ϕ on $\mathbb{K}'$.

For every $a \in H'$ and $b \in (\theta_{H'})_a$ such that $b \in H'$, we have $(a, b) \in \theta_{H'}$. So, there exists $i < \alpha$ such that $a, b \in C_i$. Therefore $(a, b) \in \phi$ and $(\theta_{H'})_a \subseteq \phi_a$. Also if $b \in \phi_a$ such that $b \in A'$, we have $(a, b) \in \phi$ and

(α) if there exists $i < \alpha$ such that $a, b \in \{b_i\}$, then $a = b = b_i$ and $a, b \notin H'$;

(β) if $a, b \in \{m\}$, then $a = b = m$ and $a, b \notin H'$;

(γ) if there exists $i < \alpha$ such that $a, b \in C_i = B_i \cap H'$, then $b \in H'$ and $(a, b) \in \theta_{H'}$. Therefore $\phi_a \subseteq (\theta_{H'})_a$. and $(\theta_{H'})_a = \phi_a$. This completes the proof of (ii). □

Corollary 3.13. *Let $\mathbb{H} = \langle H, (\beta_i| \ i \in I)\rangle$ be a fuzzy partial hyperalgebra and let θ be a congruence relation on $\mathbb{H}$. Then, there exists a fuzzy hyperalgebra $\mathbb{K}$ that contains $\mathbb{H}$ as a relative fuzzy subhyperalgebra, and there is a congruence relation ϕ on $\mathbb{K}$ such that $\phi_H = \theta$.*

Theorem 3.14. *Let $\mathbb{H} = \langle H, (\beta_i| \ i \in I)\rangle$ be a fuzzy partial hyperalgebra and θ be a congruence relation on $\mathbb{H}$. Then θ is strong if and only if $\mathbb{H}$ can be embedded into a fuzzy hyperalgebra $\mathbb{K}$ and θ can be extended to a congruence relation ρ on $\mathbb{K}$ such that for all $a \in H$, $\theta_a = \rho_a$.*

Proof. Let θ be a strong congruence relation on $\mathbb{H}$. Since $\mathbb{H}$ and $\mathbb{K}$ are their own fuzzy θ-strong subhyperalgebras, by Theorem 3.12, $\mathbb{H}$ can be embedded into the fuzzy hyperalgebra $\mathbb{K}$ and θ can be extended to congruence ρ on $\mathbb{K}$ such that $\theta_a = \rho_a$, for every $a \in H$.

Conversely, let $\mathbb{H}$ be a fuzzy relative hyperalgebra of the fuzzy hyperalgebra $\mathbb{K}$. Then $H \subseteq K$ and

$$D(\beta_i, \mathbb{H}) = \{(a_1, \ldots, a_{n_i})|\ (a_1, \ldots, a_{n_i}) \in D(\beta_i, \mathbb{K}) \cap H^{n_i}, \beta_i(a_1, \ldots, a_{n_i}) \in \mathcal{F}^*(H)\}.$$

Let ρ be a congruence relation on $\mathbb{K}$ extending the congruence relation θ on $\mathbb{H}$ such that $\theta_a = \rho_a$, for every $a \in H$. We need to show that θ is a strong congruence relation on $\mathbb{H}$, i.e., for every $a_1, \ldots, a_{n_i}, b_1, \ldots, b_{n_i} \in H$ such that $(a_1, b_1) \in \theta, \ldots, (a_{n_i}, b_{n_i}) \in \theta$, the following holds:

$$\beta_i(a_1, \ldots, a_{n_i})\ \overline{\overline{\theta}}\ \beta_i(b_1, \ldots, b_{n_i}),$$

or equivalently, for all $x, y \in H$ such that $\beta_i(a_1, \ldots, a_{n_i})(x) > 0$ and $\beta_i(b_1, \ldots, b_{n_i})(y) > 0$, $(x, y) \in \theta$. Since $a_i \in H$, $(a_1, \ldots, a_{n_i}) \in H^{n_i}$. Also, since $\mathbb{H}$ is fuzzy relative subhyperalgebra of $\mathbb{K}$, we have $(a_1, \ldots, a_{n_i}) \in D(\beta_i, \mathbb{K})$. Similarly, $(b_1, \ldots, b_{n_i}) \in D(\beta_i, \mathbb{K})$. Since ρ is a congruence relation on $\mathbb{K}$ and $a_i, b_i \in K$, if $(a_i, b_i) \in \rho$, we have

$$\beta_i(a_1, \ldots, a_{n_i})\overline{\rho}\ \beta_i(b_1, \ldots, b_{n_i}).$$

Hence for all $m \in K$, if $\beta_i(a_1, \ldots, a_{n_i})(m) > 0$, there exists $n \in K$ such that $\beta_i(b_1, \ldots, b_{n_i})(n) > 0$ and $(m, n) \in \rho$. Since $x, y \in H$, we have $x, y \in K$ and $(x, y) \in \rho$. Also since $\theta_a = \rho_a$, for every $a \in H$, we have $(x, y) \in \rho$. Therefore θ is a strong congruence relation on $\mathbb{H}$. □

Example 3.15. Let $\mathbb{H} = \langle H = \{a, b\}, \beta\rangle$ and $\mathbb{K} = \langle K = \{a, b, c\}, \beta\rangle$ be two fuzzy hyperalgebras with the binary hyperoperations defined as shown in the following tables. If $\theta = \{(a, a), (b, b), (a, b), (b, a)\}$ and $\rho = \{(a, a), (b, b), (c, c), (a, b), (b, a)\}$, then θ and ρ are the congruence relations on $\mathbb{H}$ and $\mathbb{K}$, respectively.

H \ β	$\beta(a, a)$	$\beta(a, b)$	$\beta(b, a)$	$\beta(b, b)$
a	0	0.2	0.3	0
b	0.2	0.3	0.5	0.3

K \ β	$\beta(a, a)$	$\beta(a, b)$	$\beta(a, c)$	$\beta(b, a)$	$\beta(b, b)$	$\beta(b, c)$	$\beta(c, a)$	$\beta(c, b)$	$\beta(c, c)$
a	0	0.2	0.3	0.3	0	0.4	0.1	0	0.7
b	0.2	0.3	0.5	0	0.3	0.5	0	0.2	0.2
c	0	0	0	0	0	0	0	0	0.3

Then $\mathbb{H}$ is a fuzzy partial hyperalgebra and also $\mathbb{H}$ is a fuzzy relative subhyperalgebra of the fuzzy partial hyperalgebra $\mathbb{K}$ which applies to the conditions of Theorems 3.12 and 3.14.

4. Conclusions

In this paper, the concept of fuzzy partial hyperalgebra was introduced. This concept is naturally suited for discussing subsets of hyperalgebra rather than fuzzy subhyperalgebras. If we take any subset H of fuzzy hyperalgebra $\mathbb{K}$ that is not necessarily closed under the fuzzy hyperoperations, we can delete all n_i-tuples in the domain of each n_i-ary fuzzy hyperoperation β_i that would yield a value outside of H, thus resulting in a fuzzy partial hyperalgebra and by characterizing the relationships between congruence and strong congruence, we have simplified the examination of this issue. Future research may focus on algebraic properties such as distributivity. These directions could further advance the development of hyperstructure theory.

References

[1] M. Al-Tahan and B. Davvaz, *N-ary hyperstructures associated to the genotypes of F_2-offspring*, International Journal of Biomathematics, **10** (2017), 1750118.

[2] R. Ameri and I.G. Rosenberg, *Congruences of multialgebras*, Journal of Multiple-Valued Logic and Soft Computing, **15** (2009), 525–536.

[3] R. Ameri and M.M. Zahedi, *Hyperalgebraic systems*, Italian Journal of Pure and Applied Mathematics, **6** (1999), 21–32.

[4] R. Ameri and T. Nozari, *Fuzzy hyperalgebras*, Computers and Mathematics with Applications, **61** (2011), 149–154.

[5] P. Corsini and V. Leoreanu, *Application of hyperstructure theory*, Springer, 2003.

[6] P. Corsini and I. Tofan, *On fuzzy hypergroups*, PU.M.A, **8** (1997), 29–37.

[7] B. Davvaz and I. Cristea, *Fuzzy Algebraic Hyperstructures*, Studies in Fuzziness and Soft Computing, **321** (2015), 38–46.

[8] G. Grätzer and G.H. Wenzel, *On the concept of congruence relation in partial algebras*, Mathematica Scandinavica, **20** (1967), 275–280.

[9] V. Leorenu-Fotea and B. Davvaz, *Fuzzy hyperrings*, Fuzzy Sets and Systems, **160** (2008), 2366–2378.

[10] V. Leorenu-Fotea, *Fuzzy hypermodules*, Computers and Mathematics with Applications, **57** (2009), 466–475.

[11] C. G. Massouros and G. G. Massouros, *An overview of the foundations of the hypergroup theory*, Mathematics, **9** (2021), 1014.

[12] F. Marty, *Sur une generalization de la notion de groupe*, in: 8th Congress Math. Scandinaves, Stockholm (1934), 45–49.

[13] M. Novák, *n-ary hyperstructures constructed from binary quasi-ordered semigroups*, Analele Universitatii Ovidius Constanta - Seria Matematica, **22** (2022), 147–168.

[14] D. Schweigert, *Congruence relations of multialgebras*, Discrete Mathematics, **53** (1985), 249–253.

[15] M.K. Sen, R. Ameri and G. Chowdhury, *Fuzzy hypersemigroups*, Soft Comput, **12** (2008), 891–900.

[16] A. Sonea and I. Cristea, *Euler's totient function applied to complete hypergroups*, AIMS Mathematics, **8** (2023), 7731–7746.

[17] L. A. Zadeh, *Fuzzy sets*, Information and Control, **8** (1965), 338–353.