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Research Paper

SECURE DOMINATION NUMBER OF GENERALIZED THORN GRAPHS

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ABSTRACT

A secure dominating set $S \subseteq V$ is a dominating set of G satisfying the condition that for each $u \in V \setminus S$, there exists a vertex $v \in N(u) \cap S$ such that $(S \setminus \{v\}) \bigcup \{u\}$ is a dominating set of G. The minimum cardinality of a secure dominating set of G is called the secure domination number of G, $\gamma_s(G)$. In this paper, we obtain the secure domination number of generalized thorn paths, thorn graphs, and some special graph classes like thorn rod, thorn star and Kragujevac trees, where the generalized thorn paths are important in the study of chemical compounds.

1. Introduction

The graphs discussed in this paper are finite, undirected, and simple. We recommend [4] for graph theory definitions, terminologies, and notations that are not discussed in this paper. In this context, G refers to $G = (V, E)$, where $V = V(G)$ is the vertex set and $E = E(G)$ is the edge set.

A vertex u is a neighbour of a vertex v in a graph G if uv is an edge of G. The set of all neighbours of v is the open neighbourhood of v and is denoted by $N(v)$. The set $N[v] = N(v) \cup \{v\}$ is the closed neighbourhood of v in G. The number of edges incident

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with $v \in V$ is called the degree of v in G and is denoted by $d(v)$. A pendant vertex of G is the vertex of degree one. Let $S \subseteq V$ and $v \in S$, a vertex $u \in V$ is an S-private neighbour of v if $N(u) \cap S = \{v\}$. The set of all S-private neighbours of v is denoted by $PN(v, S)$. If $u \in V \setminus S$, then u is called an S-external private neighbour of v. The set of all S-external private neighbours of v is denoted by $EPN(v, S)$ [8].

A subset S of V is a dominating set of G if every vertex $u \in V \setminus S$ has at least one neighbour in S. The minimum cardinality of a dominating set is called the domination number of G, denoted as $\gamma(G)$. A dominating set of G of cardinality $\gamma(G)$ is called a γ-set [4].

A subset $S \subseteq V$ is a secure dominating set of G if it satisfies the condition that for each $u \in V \setminus S$, there exists a vertex $v \in N(u) \cap S$ such that $(S \setminus \{v\}) \bigcup \{u\}$ is a dominating set of G. The minimum cardinality of a secure dominating set of G is called the secure domination number of G, denoted as $\gamma_s(G)$. A secure dominating set of G with cardinality $\gamma_s(G)$ is called a γ_s-set [8].

A secure dominating set of G is a collection of locations where guards can be stationed to ensure that the location complex modelled by G is protected in such a way that if a security issue arises at location u, either a guard can be stationed there to fix the issue, or a guard can fix the issue from an adjacent location, location v, and still leave the location complex dominated after relocating from location v to location u. The secure domination number, in this case, indicates the absolute minimum number of guards required to secure the entire site complex, thereby lowering the overall cost of guard deployment. The aforementioned general application is often realized in the context of military strategy analysis, surveillance applications, or the deployment of security guards by commercial security organizations [7].

E. J. Cockayne, and et.al introduced the concept of secure domination in [11], and further explored in [17] and [10]. A constructive characterization of γ-excellent trees was used by C. M. Mynhardt, and et.al in [17] to obtain a constructive characterization of trees with equal secure domination and domination numbers. In [10], Cockayne obtained a bound for a secure domination number of trees in terms of maximum degree. In [13], P. J. P. Grobler and C. M. Mynhardt studied how the edge removal affected $\gamma_s(G)$ and characterise a few classes of γ_s - ER critical graphs. In [21], D. Yun-Ping, W. Haichao, and Z. Yancai showed that the decision version of the secure domination problem is NP-complete for star convex bipartite graphs and doubly chordal graphs.

Definition 1.1. Let G be a simple connected graph of order n with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and let $P = (p_1, p_2, \ldots, p_n)$ be the n-tuple of non-negative integers. The thorn graph G_P is the graph obtained by attaching p_i pendant vertices to the vertex v_i of G for $i = 1, 2, \ldots, n$. The pendant vertices attached to the vertices v_i of G are called thorns of v_i [2].

Definition 1.2. The graph formed by attaching t copies of a path P_r to each vertex of a path P_n is referred to as a generalized thorn path and is denoted by $G(n, r, t)$, $n > 1$, $r \geq 1$, $t \geq 1$ [20].

Definition 1.3. A thorn rod, denoted as $P_{n,t}$, is formed by considering a path P_n on $n \geq 2$ vertices and attaching $(t - 1)$ pendant vertices to each of the end vertices of P_n [2]. See Figure 2.

Definition 1.4. Let $S_n = K_{1,n-1}$ denotes the star graph. Denote the vertices as $1, 2, \dots, n-1, n$, where n is the central vertex and the rest of the vertices are the terminal vertices. Then the thorn star $S_n(p_1, p_2, \dots, p_{n-1})$ is obtained by attaching p_i pendant vertices to the vertex i of S_n for $i = 1, 2, \dots, n-1$ [2]. See Figure 3.

Definition 1.5. The Kragujevac tree say T in [2] is a tree with a vertex of degree $n > 1$ (which is the central vertex of T), which is adjacent to the roots of $B_{p_1}, B_{p_2}, \dots, B_{p_n}$ where, $p_1, p_2, \dots, p_n \geq 1$. Here, n is the degree of T. The subgraphs $B_{p_1}, B_{p_2}, \dots, B_{p_n}$ are the branches of T. Kragujevac tree of degree n is usually denoted by $Kg(p_1, p_2, \dots, p_n)$ or in short Kg. Figure 4 shows Kragujevac tree $Kg(3, 2, 1, 3)$.

Gutman [14] introduced the idea of thorn graphs, which went on to uncover various chemical applications. The study of thorn graphs was motivated by the particular case below, namely $G_p = G_{\gamma-\gamma_1,\gamma-\gamma_2,\dots\gamma-\gamma_n}$ where, γ is a constant and γ_i is the degree of the i^{th} vertex of G ($\gamma_i \leq \gamma$ for all $i = 1, 2, \dots, n$). Then the degree of the vertices of G_p are either γ or one. If $\gamma = 4$, then, according to Cayley in [9], the thorn graph G_P is the plerogram (a graph in which every atom is represented by a vertex and adjacent atoms are connected by a chemical bond) and the parent graph G is the kenogram (a graph obtained from a plerogram by suppressing hydrogen atoms), and according to Polya in [19], G_P is the $C-H$ graph and G is the C-graph [2]. Later, research has focused on several kinds of thorn graphs, including thorn trees, thorn rings, thorn rods, and thorn stars. In the theory of polymers, particularly for dendrimers [3], thorn graphs were used.

Recently, there's been a lot of interest in finding secure domination in different graph classes as shown in studies like [6], [18], [15], and [1]. In [6], R. Burdett, M. Haythorpe, and A. Newcombe examined the secure domination number of flower snarks, which are a family of 3-regular graphs. In [18], P. G. Nayana and I. R. Rajamani, investigates the secure domination number of generalized Mycielskians for path graphs P_n and cycle graphs C_n. In [15], M. Haythorpe and A. Newcombe investigates the secure domination numbers of Cartesian products of small graphs with paths and cycles. In [1], R. Arasu and N. Parvathi explores the secure domination parameters of Halin graphs when combined with perfect k-ary trees.

This has inspired us to find the same in a well-known network called thorn graphs and its more general forms. In [12], we studied the secure domination number of Sierpiński graphs.

In this paper, we obtain the secure domination number of generalized thorn paths, thorn graphs G_P with $P = (p_1, p_2, \dots, p_n)$ with $p_i \neq 0$, for all i, and some special graph classes like thorn rod, thorn star in which $p_i = 0$, for some i, and Kragujevac trees, where generalized thorn paths, $G(n, r, t)$, play a significant role in the study of chemical compounds.

2. Preliminary Results

Proposition 2.1. [11] *Let S be a dominating set of G. Vertex $v \in S$ defends $u \in V \setminus S$ if and only if $G[EPN(v, S) \cup \{u, v\}]$ is complete.*

Corollary 2.2. [11] *S is a secure dominating set if and only if for each $u \in V \setminus S$, there exists $v \in S$ such that $G[EPN(v, S) \cup \{u, v\}]$ is complete.*

Theorem 2.3. [11] For $n \in \mathbb{N}$,
(1) $\gamma_s(P_n) = \lceil \frac{3n}{7} \rceil$.
(2) $\gamma_s(K_{1,n}) = n$.

Remark 2.4. The γ_s-set of P_{7m} for $m \in \mathbb{N}$, is unique [11], and the set is $S = \{7k+2, 7k+4, 7k+6 | k = 0, 1, \ldots m-1\}$.

Remark 2.5. For $m \in \mathbb{N}$,
(i) $\gamma_s(P_{7m}) = 3m$.
(ii) $\gamma_s(P_{7m-2}) = \gamma_s(P_{7m-1}) = \gamma_s(P_{7m})$.
(iii) $\gamma_s(P_{7m-3}) = \gamma_s(P_{7m})$ - 1.

Remark 2.6. In P_{7m} for $m \in \mathbb{N}$, every $v \in V \setminus S$ is uniquely S-defended by a vertex in S, or in other words, every $u \in S$ has to S-defend at least one vertex $v \in V \setminus S$, where S denotes the γ_s-set of P_{7m}.

Remark 2.7. In P_n, there exists no γ_s-set containing at least one pendant vertex if and only if $n = 7m$.

3. Secure Domination Number of Generalized Thorn Path

Throughout this section, let $1, 2, \ldots, n$ denote the vertices of P_n, and let $i1P_r$, $i2P_r$, ..., itP_r $(r \geq 1)$ denote the t copies of P_r adjacent to the vertex $i \in V(P_n)$, for $i \in \{1, 2, \ldots, n\}$ with vertex set $\{i11, i12, \ldots, i1r, i21, i22, \ldots, i2r, \ldots, it1, it2, \ldots, itr\}$.

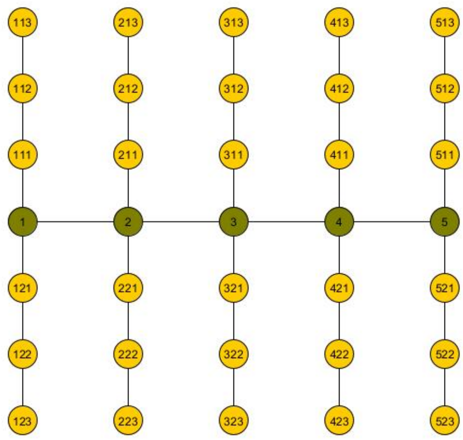

Figure 1. $G(5,3,2)$

Theorem 3.1. *For any $n, r, t \in \mathbb{N}$, $\gamma_s(G(n,r,t)) \leq nt\gamma_s(P_r) + \gamma_s(P_n)$. Moreover, the bound is attained if and only if $r = 7m$, $m \in \mathbb{N}$.*

Proof. Let S_1 and S_2 denote the γ_s-set of P_r and P_n, respectively. Define, $S = \{iju | i \in \{1, 2, \ldots, n\}, j \in \{1, 2, \ldots, t\}, u \in S_1\} \cup \{v | v \in S_2\}$, where $|S| = nt\gamma_s(P_r) + \gamma_s(P_n)$. If $x \in V(P_r)$ is adjacent to $u \in S_1$, then the vertex of the form $ijx \in V(G(n,r,t))$ is adjacent to the vertex $iju \in S$, for any i, j. Similarly, if $y \in V(P_n)$ is adjacent to $v \in S_2$, then the same $y \in V(G(n,r,t))$ is adjacent to the same $v \in S$. Hence, S is a dominating set of $G(n,r,t)$.

Since, for any $x \in V(P_r) \setminus S_1$, there exists $u \in S_1$ such that $(S_1 \setminus \{u\}) \cup \{x\}$ is a dominating set of P_r, we have for any $ijx \in V(G(n,r,t)) \setminus S$, there exists $iju \in S$ such that $(S \setminus \{iju\}) \cup \{ijx\}$ is a dominating set of $G(n,r,t)$. Therefore, S is a secure dominating set of $G(n,r,t)$. Hence, $\gamma_s(G(n,r,t)) \leq nt\gamma_s(P_r) + \gamma_s(P_n)$.

Assume that $r = 7m$, for $m \in \mathbb{N}$. The corresponding graph is $G(n, 7m, t)$. For each $i \in V(P_n)$, define $S_i = \bigcup_{j=1}^{t} \{ij7k + 2, ij7k + 4, ij7k + 6 | k = 0, 1, \ldots, m-1\}$, where $t \geq 1$, $m \in \mathbb{N}$. Let $S = \bigcup_{i=1}^{n} S_i$. Here, $|S| = nt(3m) = nt\gamma_s(P_{7m})$. Since none of the vertices of P_n is adjacent to the vertices in S, we have to choose $\gamma_s(P_n)$ number of vertices from P_n itself to form a secure dominating set of $G(n, 7m, t)$. Hence, $\gamma_s(G(n,r,t)) \leq nt\gamma_s(P_r) + \gamma_s(P_n)$. Since γ_s-set of P_{7m} is unique 2.4, a minimum of $nt\gamma_s(P_r) + \gamma_s(P_n)$ number of vertices are needed to securely dominate $G(n, 7m, t)$. Thus, $\gamma_s(G(n,r,t)) = nt\gamma_s(P_r) + \gamma_s(P_n)$.

Assume that $\gamma_s(G(n,r,t)) = nt\gamma_s(P_r) + \gamma_s(P_n)$. Here, P_n is securely dominated by $\gamma_s(P_n)$ number of vertices from P_n itself. This implies that no vertices of P_r are present to dominate the vertices of P_n. i.e., no pendant vertex of P_r is in the γ_s-set of P_r. This scenario is possible only if $r = 7m$, $m \in \mathbb{N}$, refer Remark 2.7. □

In the upcoming theorems, we aim to determine the exact values for $\gamma_s(G(n,r,t)$ based on the values of r in the paths P_r. Since the γ_s-set of P_{7m} for $m \in \mathbb{N}$ is unique, we consider the values of r as $r = 7m + k$, for $k = 0, 1, 2, 3, 4, 5, 6$. According to Theorem 3.1, we have $\gamma_s(G(n,r,t)) = nt\gamma_s(P_r) + \gamma_s(P_n)$ if and only if $r = 7m$, where $m \in \mathbb{N} \cup \{0\}$. The theorems below provides the $\gamma_s(G(n,r,t))$ for the remaining values of r.

Theorem 3.2. *For any $n, r \in \mathbb{N}$, $t > 1$, $\gamma_s(G(n,r,t)) = nt\gamma_s(P_r) + \gamma(P_n)$ if and only if $r = 7m + 2$, $m \in \mathbb{N} \cup \{0\}$.*

Proof. Assume that $r = 7m+2$, where $m \in \mathbb{N} \cup \{0\}$. The corresponding graph is $G(n, 7m+2, t)$. Define $S_i = \bigcup_{j=1}^{t} \{ij7k+2, ij7k+4, ij7k+6, ij7m+2 | k = 0, 1, \ldots, m-1\}$. Then, define $S = \bigcup_{i-1}^{n} S_i$. Here, $|S| = nt\gamma_s(P_r)$, for $r = 7m + 2$. Although the vertex $ij7m + 2$ S-defends the vertex $ij7m + 1$, it can also dominate another adjacent vertex, likely the vertex i of P_n. Since $t > 1$, for each vertex $i \in V(P_n)$, there exist t vertices $i17m+2, i27m+2, \ldots, it7m+2$ which can dominate i. Therefore, only $\gamma(P_n)$ vertices are needed to securely dominate the vertices of P_n. Thus, we have $\gamma_s(G(n,r,t)) \leq nt\gamma_s(P_r) + \gamma(P_n)$.
Suppose S' is the γ_s-set of $G(n, 7m+2, t)$. Since each P_r is disjoint, we have $|S'| \geq nt\gamma_s(P_r)$ for $r = 7m + 2$. In order to form a secure dominating set of $G(n, 7m + 2, t)$, a minimum of $\gamma(P_n)$ vertices are required from P_n. Thus, $|S'| \geq nt\gamma_s(P_r) + \gamma(P_n)$. Hence, $\gamma_s(G(n,r,t)) = nt\gamma_s(P_r) + \gamma(P_n)$.

Assume that $\gamma_s(G(n,r,t)) = nt\gamma_s(P_r) + \gamma(P_n)$. Here, $\gamma_s(P_r)$ vertices are taken from each copy of P_r and $\gamma(P_n)$ vertices from P_n. For P_n, $\gamma(P_n)$ vertices are enough to securely dominates the vertices of P_n. This happens only when each vertex $i \in V(P_n)$ is dominated by at least on vertex of P_r. Additionally, these vertices do not S-defend the vertex i, as $\gamma(P_n)$ vertices are added to form a secure dominating set of $G(n,r,t)$. This is possible only if $r = 7m + 2$. □

Theorem 3.3. *For any $n \in \mathbb{N}$, $t > 1$, $\gamma_s(G(n,r,t)) = nt\gamma_s(P_r) - n(t-1)$ if and only if $r = 7m + k$, for $k = 3, 5$, $m \in \mathbb{N} \cup \{0\}$.*

Proof. Assume that $r = 7m + k$, for $k = 3, 5$, $m \in \mathbb{N} \cup \{0\}$.
Case 1: $r = 7m + 3$.

The corresponding graph is $G(n, 7m+3, t)$. Define
$S_i = \bigcup_{j=1}^{t}\{ij7k+2, ij7k+4, ij7k+6, ij7m+2, ij7m+3 | k = 0, 1, \ldots, m-1\}$. Let
$S' = \bigcup_{i=1}^{n} S_i$. Here, $|S'| = nt\gamma_s(P_r)$, for $r = 7m+3$. Define
$S'' = \bigcup_{j=1}^{t}\{ij7m+3 | i = 1, 2, \ldots, n\}$. Consider $S = (S' \setminus S'') \cup V(P_n)$. We are going to show that S forms a γ_s-set of $G(n, 7m+3, t)$. It is enough to show that the vertices of S'' are securely dominated by the vertices of S.

Clearly, S is a dominating set of $G(n, 7m+3, t)$. Consider a vertex, say $ij7m+3$, for $i \in \{1, 2, \ldots, n\}$ and $j \in \{1, 2, \ldots, t\}$ where $t > 1$. Here, $ij7m+3 \in S''$ is S-defended by the vertex $i \in V(P_n)$ for all j. Also, $ij7m+2$ dominates $ij7m+3$. Hence, S is a secure dominating set of $G(n, 7m+3, t)$. Since $t > 1$, S is the γ_s-set of $G(n, 7m+3, t)$. Hence,

$$\begin{aligned}\gamma_s(G(n, r, t)) &= nt(\gamma_s(P_r) - 1) + n \\ &= nt\gamma_s(P_r) - n(t-1).\end{aligned}$$

Case 2: $r = 7m+5$.
The corresponding graph is $G(n, 7m+5, t)$. Define
$S_i = \bigcup_{j=1}^{t}\{ij7k+2, ij7k+4, ij7k+6, ij7m+2, ij7m+4, ij7m+5 | k = 0, 1, \ldots, m-1\}$. Let
$S' = \bigcup_{i=1}^{n} S_i$. Here, $|S'| = nt\gamma_s(P_r)$, for $r = 7m+5$. Define
$S'' = \bigcup_{j=1}^{t}\{ij7m+5 | i = 1, 2, \ldots, n\}$. Consider $S = (S' \setminus S'') \cup V(P_n)$. We are going to show that S forms a γ_s-set of $G(n, 7m+5, t)$. It is enough to show that the vertices of S'' are securely dominated by the vertices of S.

Clearly, S is a dominating set of $G(n, 7m+5, t)$. Consider a vertex $ij7m+5 \in S''$ for $i \in \{1, 2, \ldots, n\}$ and $j \in \{1, 2, \ldots, t\}$, where $t > 1$. Here, $ij7m+5$ is S-defended by the vertex $i \in V(P_n)$ for all j. Also, $ij7m+4$ dominates $ij7m+5$. Hence, S is a secure dominating set of $G(n, 7m+5, t)$. Since $t > 1$, S is the γ_s-set of $G(n, 7m+5, t)$. Hence,

$$\begin{aligned}\gamma_s(G(n, r, t)) &= nt(\gamma_s(P_r) - 1) + n \\ &= nt\gamma_s(P_r) - n(t-1).\end{aligned}$$

Assume that $\gamma_s(G(n, r, t)) = nt\gamma_s(P_r) - n(t-1) = nt(\gamma_s(P_r) - 1) + n$. From Theorem 3.1 and Theorem 3.2, we get $n \neq 7m+k$ for $k = 0, 2$. We can easily conclude that the result holds true only if $k = 3, 5$. □

Theorem 3.4. *For any $n \in \mathbb{N}$, $t > 1$, $\gamma_s(G(n, r, t)) = nt\gamma_s(P_r)$ if and only if $r = 7m+k$, for $k = 1, 4, 6$, $m \in \mathbb{N} \cup \{0\}$.*

Proof. Assume that $r = 7m+k$, for $k = 1, 4, 6$, $m \in \mathbb{N} \cup \{0\}$.
Case 1: $r = 7m+1$
The corresponding graph is $G(n, 7m+1, t)$. Define $S_i = \bigcup_{j=1}^{t}\{ij7k+2, ij7k+4, ij7k+6, ij7m+1 | k = 0, 1, \ldots, m-1\}$. Let $S = \bigcup_{i=1}^{n} S_i$. Here, $|S| = nt\gamma_s(P_r)$, for $r = 7m+1$. We are going to show that S forms a γ_s-set of $G(n, r, t)$ for $r = 7m+1$.

From the selection of vertices in S, it is clear that S securely dominates all the vertices of ijP_{7m+1} for all i and j. It is enough to show that S securely dominates the vertices of P_n. Consider a vertex $i \in V(P_n)$. Any vertex of the form $ij7m+1$ where $j \in \{1, 2, \ldots, t\}$ securely dominates i. Hence, S is a secure dominating set of $G(n, r, t)$ for $r = 7m+1$.

Let S' denote the γ_s-set of $G(n, r, t)$. We have to show that $|S| \leq |S'|$. Since there are no edges connecting the paths ijP_{7m+1}, $|S'| \geq nt\gamma_s(P_{7m+1})$, $m \in \mathbb{N} \cup \{0\}$. Hence, $|S| \leq |S'|$.

Thus S forms the γ_s-set of $G(n, 7m+1, t)$.

Case 2: $r = 7m+4$

The corresponding graph is $G(n, 7m+4, t)$. Define $S_i = \bigcup_{j=1}^{t}\{ij7k+2, ij7k+4, ij7k+6, ij7m+2, ij7m+4 | k = 0, 1, \dots, m-1\}$. Let S'$=\bigcup_{i=1}^{n} S_i$. Here, $|S'| = nt\gamma_s(P_r)$, for $r = 7m+4$. We are going to show that S forms a γ_s-set of $G(n, r, t)$ for $r = 7m+4$.

From the selection of vertices in S, it is clear that S securely dominates all the vertices of ijP_{7m+4} for all i and j. It is enough to show that S securely dominates the vertices of P_n. Consider a vertex $i \in V(P_n)$. Any vertex of the form $ij7m+4$ where $j \in \{1, 2, \dots, t\}$ securely dominates i. Hence, S is a secure dominating set of $G(n, 7m+4, t)$.

Let S' denotes the γ_s-set of $G(n, 7m+4, t)$. We have to show that $|S| \leq |S'|$. Since there are no edges connecting the paths ijP_{7m+4}, $|S'| \geq nt\gamma_s(P_{7m+4})$, $m \in \mathbb{N} \cup \{0\}$. Hence, $|S| \leq |S$'$|$. Thus, S forms the γ_s-set of $G(n, 7m+4, t)$.

Case 3: $r = 7m+6$

The corresponding graph is $G(n, 7m+6, t)$. Define $S_i = \bigcup_{j=1}^{t}\{ij7k+2, ij7k+4, ij7k+6, ij7m+2, ij7m+4, ij7m+6 | k = 0, 1, \dots, m-1\}$. Let $S'=\bigcup_{i=1}^{n} S_i$. Here, $|S'| = nt\gamma_s(P_r)$, for $r = 7m+6$. We are going to show that S forms a γ_s-set of $G(n, r, t)$ for $r = 7m+6$.

From the selection of vertices in S, it is clear that S securely dominates all the vertices of ijP_{7m+6} for all i and j. It is enough to show that S securely dominates the vertices of P_n. Consider a vertex $i \in V(P_n)$. Any vertex of the form $ij7m+6$ where $j \in \{1, 2, \dots, t\}$ securely dominates i. Hence, S is a secure dominating set of $G(n, 7m+6, t)$.

Let S' denote the γ_s-set of $G(n, 7m+6, t)$. We have to show that $|S| \leq |S'|$. Since there does not contain any edge connecting the paths ijP_{7m+6}, $|S'| \geq nt\gamma_s(P_{7m+6})$, $m \in \mathbb{N} \cup \{0\}$. Hence, $|S| \leq |S'|$. Thus, S forms the γ_s-set of $G(n, 7m+6, t)$.

Assume that $\gamma_s(G(n, r, t)) = nt\gamma_s(P_r)$. We can easily conclude that the result is true only if $k = 1, 4, 6$. □

4. Secure domination number of thorn graphs

In this section, we are considering thorn graphs G_P with $P = (p_1, p_2, \dots, p_n)$, such that $p_i \neq 0$, for all i.

Theorem 4.1. *Let G_P be the thorn graph of G with $P = (p_1, p_2, \dots, p_n)$ as the n-tuple of positive integers. Then $\gamma_s(G_P) = \sum_{i=1}^{n} p_i$.*

Proof. To simplify the notation, let the vertices of G be $1, 2, \dots, n$ and let the pendant vertices adjacent to each $i \in G$ be $i1, i2, \dots, ip_i$, for $i = 1, 2, \dots, n$. Define $S = \{i1, i2, \dots, ip_i | i = 1, 2, \dots, n\}$, where $|S| = \sum_{i=1}^{n} p_i$. In other words, S is the set of all pendant vertices adjacent to G. We need to show that S forms a secure dominating set of G. Since each vertex of $V(G_P) \setminus S$ is adjacent to at least one vertex of S, S form a dominating set of G_P.

Consider any $v \in V(G_P) \setminus S$. Then, $v = i$ for some $i \in \{1, 2, \dots, n\}$. We know that i is adjacent to p_i pendant vertices $i1, i2, \dots, ip_i$ for $i = 1, 2, \dots, n$, and each of these p_i vertices belongs to S. Hence, the vertices ij, for $j = 1, 2, \dots, p_i$, S-defend the vertex i, for $i = \{1, 2, \dots, n\}$. Therefore, S is a secure dominating set of G_P. Since each vertex in S is non-adjacent, S is minimal.

$$\gamma_s(G_P) \leq \sum_{i=1}^{n} p_i. \tag{4.1}$$

Now, we need to show that S is a γ_s-set of G_P. Assume that there exists a secure dominating set S' with $|S'| \leq |S|$. Then, S' does not include all the pendant vertices in S. Since S is minimal, S' contain at least one vertex from G, say j, for $j = 1, 2, \ldots, n$.

Case 1: At least two pendant vertices are not in S'.

Suppose these two pendant vertices are of the form jr and jm for $1 \leq r, m \leq p_j$. As jr and jm are two non-adjacent external private neighbours of j, the vertex j cannot S-defend both jr and jm. Since S' is a γ_s-set of G_P, either $jr \in S'$ or $jm \in S'$. Therefore, this case is not possible.

If these two pendant vertices are of the form ir and jm for $1 \leq r \leq p_i$, $1 \leq m \leq p_j$, with $i \neq j$, then, both i and j belong to S'. Since ir and jm are external private neighbour of i and j respectively, both i and j are unable to S-defend any other vertices in G_P. Hence, $|S'| = |S|$.

Case 2: Exactly one of the p_j pendant vertices adjacent to j is not in S'.

Let jr be the pendant vertex adjacent to j, for $1 \leq r \leq p_j$. Since $jr \notin S'$, the vertex $j \in S'$. Given that jr is the external private neighbour of j, the vertex j can only S-defend jr. Therefore, from each set $\{j, j1, j2, \ldots, jp_j\}$, S' contains at least j number of vertices, for $j = 1, 2, \ldots, n$. Thus, $|S'| \geq |S|$.

Case 3: For some j, if the set $\{j, j1, j2, \ldots, jp_j\} \subseteq S'$.

Then, j can S-defend at least one adjacent vertex say i in G. Thus, $\{i1, i2, \ldots, ip_i\} \subseteq S'$. i. e., all pendant vertices adjacent to i and j belong to S'. If j S-defends all the vertices of G (when $G = K_n$), then $\cup_{i=1}^{n}\{i1, i2, \ldots, ip_i\} \cup \{j\} \subseteq S'$, which implies $|S'| > |S|$. Since S' is a γ_s-set of G_P, $|S'| = |S|$. Hence, S is a γ_s-set of G_P.

□

5. Secure domination number of some special graphs

In this section, we have obtained the secure domination number of some thorn graphs with $p_i = 0$, for some i. Here, we are considering some special graphs like thorn rod, thorn star, and Kragujevac trees.

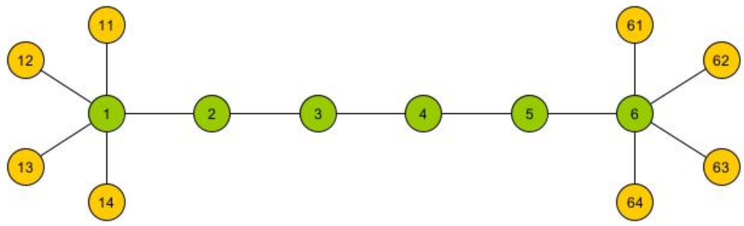

Figure 2. $P_{6,5}$

Theorem 5.1. *For $n \in \mathbb{N}$, $t \geq 1$,*

$$\gamma_s(P_{n,t}) = \begin{cases} \left\lceil \frac{3n}{7} \right\rceil + 2t - 4, & \text{if } n = 7m + 5 \\ \left\lceil \frac{3n}{7} \right\rceil + 2t - 3, & \text{Otherwise} \end{cases}$$

for $m \in \mathbb{N} \cup \{0\}$.

Proof. Let P_n be a path with n vertices, labelled as $1, 2, \ldots n$. The thorn rod $P_{n,t}$ is the graph obtained by attaching $(t-1)$ pendant vertices to vertices 1 and n, respectively. Let the $(t-1)$ pendant vertices adjacent to 1 are labelled as $11, 12, \ldots 1(t-1)$, and those adjacent to the vertex n are labelled as $n1, n2, \ldots n(t-1)$. Let S denote the γ_s-set of $P_{n,t}$.

Case 1: $\lceil \frac{3n}{7} \rceil + 2t - 4 \leq \gamma_s[P_{n,t}] \leq \lceil \frac{3n}{7} \rceil + 2t - 3$.

Since the subgraph induced by the vertices $1, 11, 12, \ldots 1(t-1)$ forms a star, at least $(t-1)$ of these vertices belong to S. Let these vertices be $1, 11, 12, \ldots 1(t-2)$. Similarly, the vertices $n, n1, n2, \ldots n(t-2)$ belong to S. Clearly, the vertices 1 and n dominate the vertices 2 and $(n-1)$ respectively. Next, the vertices 3 and $(n-2)$ belong to S, where 3 securely dominates 2 and 4, and $(n-2)$ securely dominates both $(n-1)$ and $(n-3)$. Proceeding like this, we get 5 and $(n-4)$ belong to S, where 5 securely dominates 6 and $(n-4)$ securely dominates $(n-5)$. Now, the rest of the graph is P_{n-12}. And we know that,

$$\begin{aligned} \gamma_s\{P_{n-12}\} &= \left\lceil \frac{3(n-12)}{7} \right\rceil \\ &= \left\lceil \frac{3n}{7} - \frac{36}{7} \right\rceil . \end{aligned}$$

We know that $\lceil \frac{3n}{7} \rceil - 6 \leq \lceil \frac{3n}{7} - \frac{36}{7} \rceil \leq \lceil \frac{3n}{7} \rceil - 5$. And $\gamma_s\{P_{n,t}\} = (t-1) + (t-1) + 4 + \gamma_s\{P_{n-12}\}$.
Therefore, $\lceil \frac{3n}{7} \rceil - 6 + (t-1) + (t-1) + 4 \leq \gamma_s\{P_{n,t}\} \leq \lceil \frac{3n}{7} \rceil - 5 + (t-1) + (t-1) + 4$.
Thus $\lceil \frac{3n}{7} \rceil + 2t - 4 \leq \gamma_s\{P_{n,t}\} \leq \lceil \frac{3n}{7} \rceil + 2t - 3$.

Case 2: $\gamma_s\{P_{n,t}\} = \lceil \frac{3n}{7} \rceil + 2t - 4$ if and only if $n = 7m + 5$, for $m \in \mathbb{N} \cup \{0\}$.

Since $\gamma_s\{P_{n,t}\} = (t-1) + (t-1) + 4 + \gamma_s\{P_{n-12}\}$, for $n = 7m + 5$, we get, $\gamma_s\{P_{7m+5,t}\} = (t-1) + (t-1) + 4 + \gamma_s\{P_{7m+5-12}\}$. Here,

$$\begin{aligned} \gamma_s\{P_{7m+5-12}\} &= \left\lceil \frac{3(7m-7)}{7} \right\rceil \\ &= 3m - 3. \end{aligned}$$

For $m \in \mathbb{N} \cup \{0\}$. Therefore, the LHS is given by,

$$\begin{aligned} \gamma_s\{P_{n,t}\} &= (t-1) + (t-1) + 4 + \gamma_s\{P_{n-12}\} \\ &= 2t + 2 + 3m - 3 \\ &= 3m + 2t - 1. \end{aligned}$$

$$\begin{aligned} RHS &= \left\lceil \frac{3n}{7} \right\rceil + 2t - 4 \\ &= \left\lceil \frac{3(7m+5)}{7} \right\rceil + 2t - 4 \\ &= \left\lceil 3m + \frac{15}{7} \right\rceil + 2t - 4 \\ &= 3m + 3 + 2t - 4 \\ &= 3m + 2t - 1. \end{aligned}$$

Hence, if $n = 7m + 5$, we get $\gamma_s\{P_{n,t}\} = \lceil \frac{3n}{7} \rceil + 2t - 4$.
Assume the converse. Let $n = 7m + x$, where x can be $0, 1, 2, 3, 4, 5, 6$. We need to show that LHS=RHS only if $x = 5$. Similar to the previous case, the vertices $1, 11, 12, \ldots, 1(t-2)$, $n, n1, n2, \ldots n(t-2)$, $3, 5, (n-2), (n-4)$ belong to S. The rest of the graph is $P_{n-12} =$

$P_{7m+x-12}$. Therefore,

$$\gamma_s(P_{7m+x-12}) = \left\lceil \frac{3(7m+x-12)}{7} \right\rceil = \left\lceil 3m + \frac{3(x-12)}{7} \right\rceil$$

Hence,

$$\begin{aligned} RHS &= \gamma_s(P_{7m+x,t}) \\ &= \gamma_s(P_{7m+x-12} + 2t + 2 \\ &= \left\lceil 3m + \frac{3(x-12)}{7} \right\rceil + 2t + 2 \\ &= 3m + 2t + \left\lceil \frac{3(x-12)}{7} \right\rceil + 2 \end{aligned}$$

$$\begin{aligned} LHS &= \left\lceil \frac{3n}{7} \right\rceil + 2t - 4 \\ &= \left\lceil \frac{3(7m+x)}{7} \right\rceil + 2t - 4 \\ &= \left\lceil 3m + \frac{3x}{7} \right\rceil + 2t - 4 \\ &= 3m + 2t + \left\lceil \frac{3x}{7} \right\rceil - 4 \end{aligned}$$

Now, LHS=RHS implies $3m + 2t + \left\lceil \frac{3(x-12)}{7} \right\rceil + 2 = 3m + 2t + \left\lceil \frac{3x}{7} \right\rceil - 4$.
i.e., $\left\lceil \frac{3(x-12)}{7} \right\rceil + 2 = \left\lceil \frac{3x}{7} \right\rceil - 4$.
Thus,

$$\left\lceil \frac{3x}{7} \right\rceil - \left\lceil \frac{3x-12}{7} \right\rceil = 6. \tag{5.1}$$

Substituting $x = 0, 1, 2, 3, 4, 5,$ and 6, we get, only 5 satisfies (5.1). Therefore, $\gamma_s\{P_{n,t}\} = \left\lceil \frac{3n}{7} \right\rceil + 2t - 4$ if and only if $n = 7m + 5$, for $m \in \mathbb{N} \cup \{0\}$. □

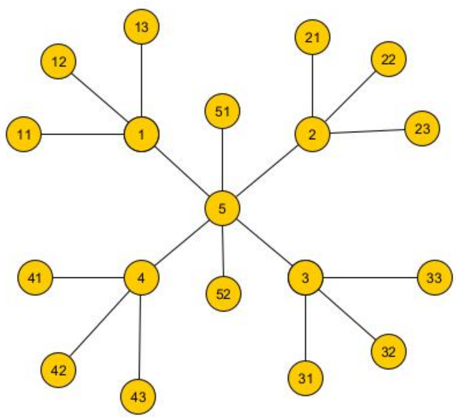

FIGURE 3. $S_5(3, 2, 1, 3)$

Theorem 5.2. *Let S_n be the star graph with n vertices. Then,* $\gamma_s(S_n(p_1, p_2, \ldots, p_{n-1})) = \sum_{i=1}^{n-1} p_i + 1$.

Proof. Consider the vertices of S_n to be $1, 2, \ldots, n$, where n is the vertex of degree $n-1$. Define, $S = \cup_{i=1}^{n-1}\{i1, i2, \ldots, ip_i\} \cup \{n\}$, where $|S| = \sum_{i=1}^{n-1} p_i + 1$. The vertices of $V(S_n(p_1, p_2, \ldots, p_{n-1})) \setminus S$ are of the form i, for $i = 1, 2, \ldots, n-1$, and each i is adjacent to p_i pendant vertices in S. Hence, S is a dominating set of $S_n(p_1, p_2, \ldots, p_{n-1})$. Furthermore, any vertex i is S-defended by any of its p_i pendant vertices in S. Thus, S forms a secure dominating set of $S_n(p_1, p_2, \ldots, p_{n-1})$. Since each vertex in S is non-adjacent, S is minimal.

Suppose there exists a γ_s-set of $S_n(p_1, p_2, \ldots, p_{n-1})$, denoted as S', with $|S'| \leq |S|$. For each $i \in \{1, 2, \ldots, n-1\}$, since the graph induced by the set of vertices $\{i, i1, i2, \ldots, ip_i\}$ is a star, S' contains at least p_i vertices from each set. That is, $|S'| \geq \sum_{i=1}^{n-1} p_i$.

Suppose S' contains exactly p_i vertices from each set $\{i, i1, i2, \ldots, ip_i\}$. In that case, none of these vertices S-defends the vertex n. Since S' is a γ_s-set of $S_n(p_1, p_2, \ldots, p_{n-1})$, either $n \in S'$ or any vertex adjacent to n belongs to S'. Consequently, $|S'| \geq \sum_{i=1}^{n-1} p_i$. Therefore, $|S'| \geq |S|$. Hence, S is a γ_s-set of $S_n(p_1, p_2, \ldots, p_{n-1})$. □

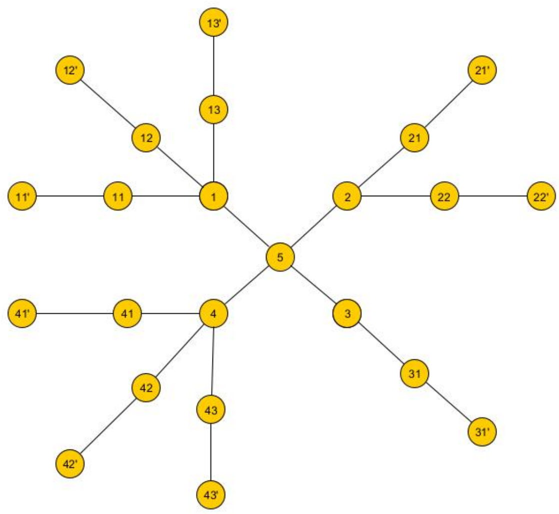

FIGURE 4. $Kg(3, 2, 1, 3)$

Lemma 5.3. $\gamma(Kg) = \sum_{i=1}^{n} p_i + 1$.

Proof. Let the central vertex of Kg be u_{n+1}, and the vertices adjacent to u_{n+1} be $u_1, u_2, \ldots, u_n$. The vertices adjacent to u_i are $u_{i1}, u_{i2}, \ldots, u_{ip_i}$ for $i = 1, 2, \ldots, n$, and the vertices adjacent to u_{ij} are denoted as u'_{ij} for $j = 1, 2, \ldots, p_i$ and $i = 1, 2, \ldots, n$. Define $S = \{u_{ij} | j = 1, 2, \ldots, p_i, i = 1, 2, \ldots, n\} \cup \{u_{n+1}\}$. Here, $V(Kg) \setminus S = \{u_1, u_2, \ldots, u_n,\} \cup \{u'_{ij} | i = 1, 2, \ldots, n, j = 1, 2, \ldots, p_i\}$.

From the figure itself, it is clear that any vertex in $V(Kg) \setminus S$ is adjacent to at least one vertex in S. Thus, S forms a dominating set of Kg. Since all the vertices in S are non-adjacent, S is minimal. We have to show that S is a γ-set of Kg.

Suppose S' is a γ-set of Kg. Since u'_{ij} is a pendant vertex adjacent to u_{ij}, either u'_{ij} or u_{ij} belongs to S' for $j = 1, 2, \ldots, p_i$ and $i = 1, 2, \ldots, n$. Hence, $|S'| \geq \sum_{i=1}^{n} p_i$. Since neither u_{ij} nor u'_{ij} dominates the central vertex u_{n+1}, at least one more vertex is needed to form a dominating set of Kg. Since S' is a dominating set of Kg, $|S'| \geq \sum_{i=1}^{n} p_i + 1$. Thus, $|S'| \geq |S|$. Hence, S is a γ-set of Kg.

$\square$

Theorem 5.4. $\gamma_s(Kg) = \sum_{i=1}^{n} p_i + 1$.

Proof. From Lemma 5.3, we have, $\gamma(Kg) = \sum_{i=1}^{n} p_i + 1$. Define $S = \{u_{ij} | j = 1, 2, \ldots, p_i, i = 1, 2, \ldots, n\} \cup \{u_{n+1}\}$, which is the same as defined in the above proof. Clearly, S is a dominating set of Kg. We need to show that S forms a secure dominating set of Kg.

Here, $V(Kg) \setminus S = \{u_1, u_2, \ldots, u_n,\} \cup \{u'_{ij} | i = 1, 2, \ldots, n, j = 1, 2, \ldots, p_i\}$. Each vertex of the form u_i is S-defended by the central vertex u_{n+1} for all $i = 1, 2, \ldots, n$. Additionally, each vertex of the form u'_{ij} is S-defended by the vertex u_{ij} for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, p_i$. Hence, S is a secure dominating set of Kg. Since S is a γ-set of Kg, S form a γ_s-set of Kg. Therefore, $\gamma(Kg) = |S| = \sum_{i=1}^{n} p_i + 1$. $\square$

6. Conclusions

Secure domination ensures that every node in the network is either secure or has a neighbouring secure node. This enhances network resilience against attacks, failures, or disruptions, ensuring continuous operation and data integrity. It is particularly important for critical infrastructure such as communication networks, financial systems, and health care networks. Since the secure domination problem is NP-complete, it is relevant to find the same in well-known networks.

Thorn graphs are well-known networks in which an atom is represented by each vertex and a chemical relationship between these atoms is represented by each edge. The present work derives the secure domination number of thorn graphs and generalized thorn paths, which have wide applications in the fields of medicine and chemistry by providing a way to visualize and analyze the molecular framework.

This kind of research has many applications because secure domination provides insights into the structural properties of the network, enabling the design of more robust and efficient network topologies. This is particularly valuable in the planning and development of new networks, ensuring that they are built to withstand potential threats and challenges. Hence, secure domination is undoubtedly a desirable feature for interconnection networks in the present day. We can expand this study by examining various other generalized versions of thorn graphs available in the literature. Instead of using paths as the base graph, we can choose different graphs to create generalized thorn graphs and explore secure domination in those configurations.

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